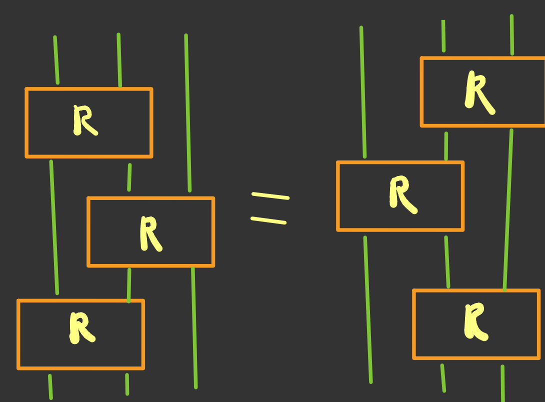
The Loop Hecke Algebra &

Charge Conserving Yang-Baxter Operators

Moduli Spaces & Friends

Eric Rowell



All Joint Work with P. Martin (2112.04533)

+ C. Damiani (Pac. J. Math. to appear)

+ F. Torzewska (2301.13831)

Motivation

2D

(2+1)D TQFT $\rightsquigarrow$ Reps of (surface) Braid gps

2D Topological Phases of Matter $\leftrightsquigarrow$ Topol. invariants of Links / 3-mflds etc.

3D

(3+1)D TQFT $\rightsquigarrow$ Reps of Motion Groups.

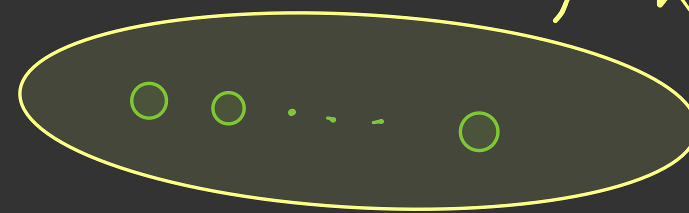
$\boxed{??}$

Abstractly, B_n is generated by $\sigma_1, \dots, \sigma_{n-1}$ satisfying:

The ***braid relations:***

(B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
(B2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for* $|i-j| > 1$

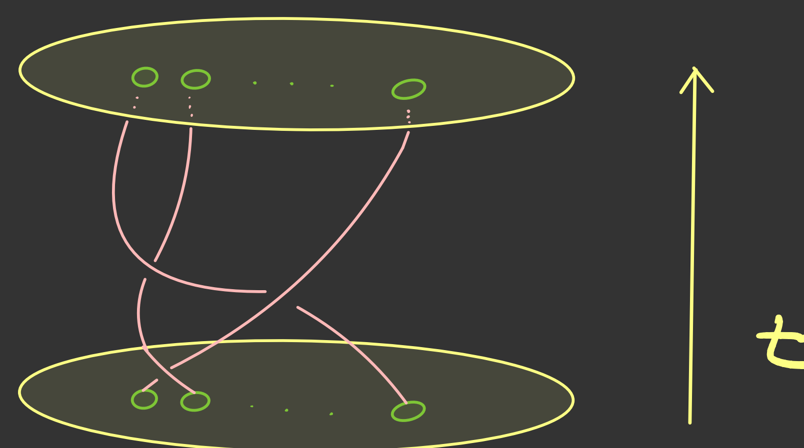
also: MCG of $\Gamma_{n+1} = D^2 \setminus \{z_1, \dots, z_n\}$



Acts on $\pi_1(\Gamma_{n+1})$ a free gp.

Geometric braids / isotopy ~

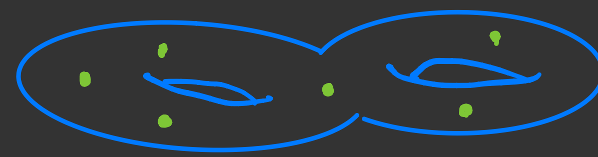
& $\boxed{\text{Motion gp}}$ of $\{z_1, \dots, z_n\} \subseteq D^2$



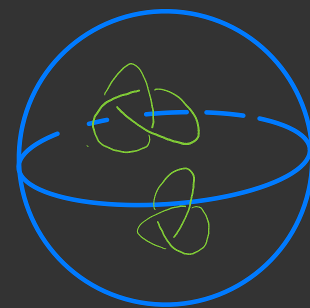
Motion Groups

Dahm '62, Goldsmith '80s

$N \subseteq M$ submanifold



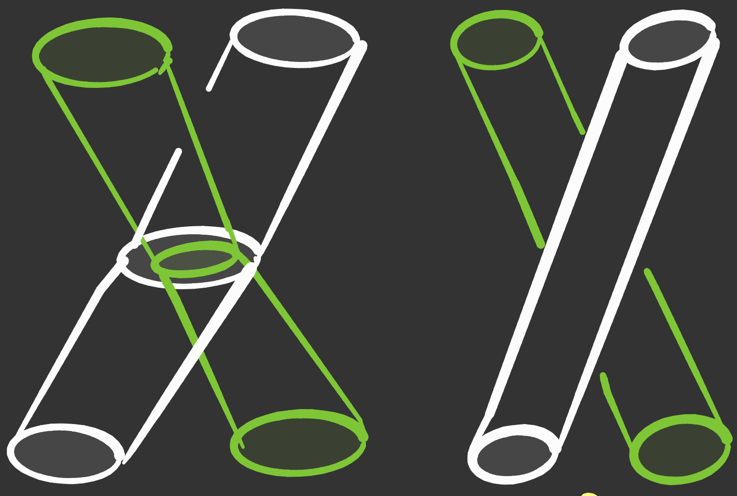
- A motion of N in M is an ambient isotopy $f_t(x)$ of N in M s.t.



1. $f_0 = \mathrm{Id}_M$
2. $f_1(N) = N$ as a submanifold

- f is stationary if $f_t(N) = N \ \forall t$
- $f \simeq g$ if $g^{-1} \circ f \sim$ a stationary motion. ($\sim$: homotopic)
- $\mathcal{M}(M, N)$ motions $/ \simeq$.

Example (McCool, Fenn-O'Roarke-Rimanyi, ...)



Loop Braid group

The ***braid relations:***

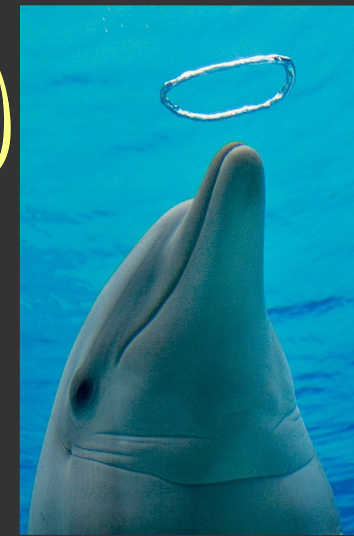
(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j|>1$,

the ***symmetric group relations:***

(S1) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$
(S2) $s_is_j = s_js_i$ *for* $|i-j|>1$,
(S3) $s_i^2 = 1$

and the ***mixed relations:***

(L0) $\sigma_is_j = s_j\sigma_i$ *for* $|i-j|>1$
(L1) $s_is_{i+1}\sigma_i = \sigma_{i+1}s_is_{i+1}$
(L2) $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}$



$$LB_n = \mathcal{M}(D^3, \sqcup^n S^1) \boxed{\supseteq B_n}$$

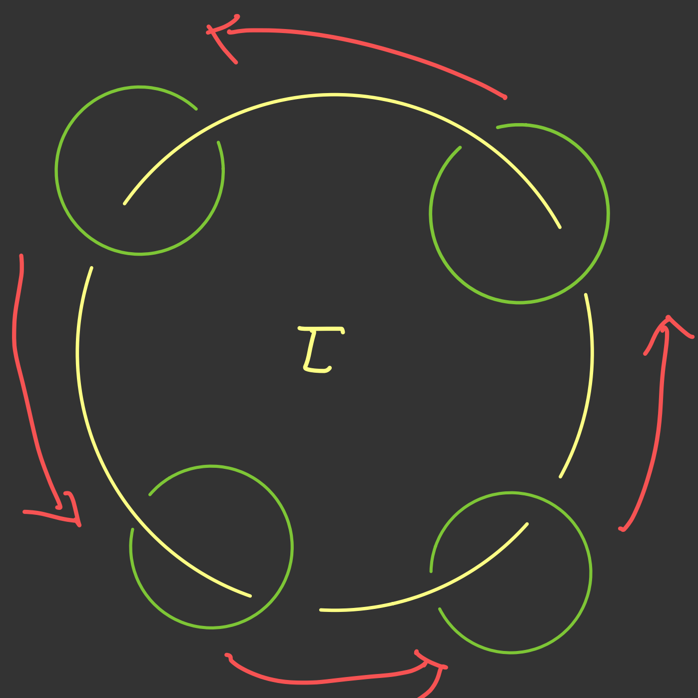
heuristic defn

Example (Bellingeri–Bodin)

$$\mathcal{M}(S^3, N) = NB_n \boxed{\supseteq B_n}$$

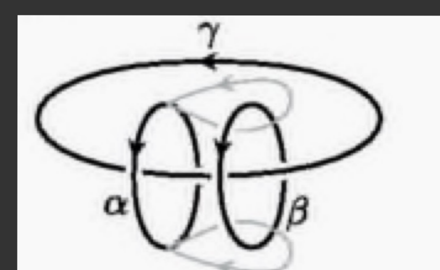
(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j| \neq 1 \pmod{n}$,
(N1) $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$ *for* $1 \leq i \leq n$
(N2) $\tau^{2n} = 1$

Here indices are taken modulo n, with $\sigma_{n+1} := \sigma_1$ *and* $\sigma_0 := \sigma_n$.

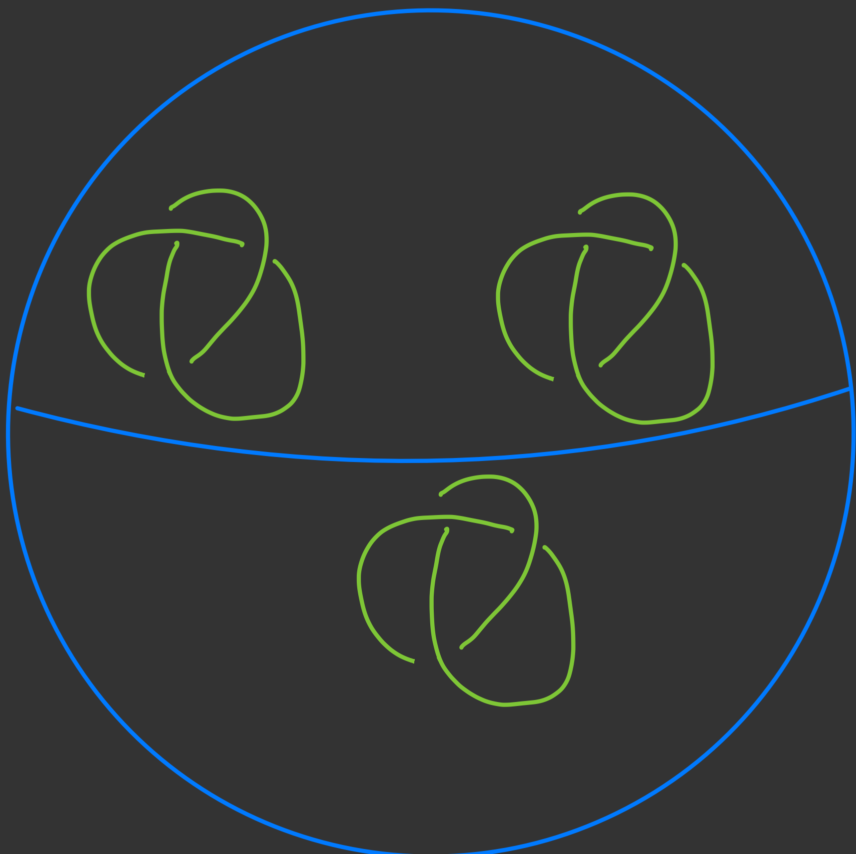


Necklace Braid Group

Levin–Wang PRL '14



More examples:



Free trefoils in

D^3

Torus Links

Problem: How to obtain reps of $\mathcal{L}B_n$?

The ***braid relations****:*

(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j|>1$,

the ***symmetric group relations****:*

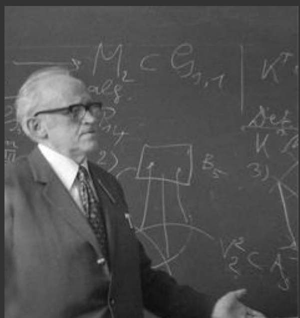
(S1) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$
(S2) $s_is_j = s_js_i$ *for* $|i-j|>1$,
(S3) $s_i^2 = 1$

and the ***mixed relations****:*

(L0) $\sigma_is_j = s_j\sigma_i$ *for* $|i-j|>1$
(L1) $s_is_{i+1}\sigma_i = \sigma_{i+1}s_is_{i+1}$
(L2) $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}$

Take some hints from B_n...

B_n representation families



Burau $\boxed{1936}$

$$\sigma_i \longmapsto \begin{bmatrix} I & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & I \end{bmatrix}$$

depending on t, $n-1$ or $n-2$ dim'l irred. subreps.

"standard rep"

$$\sigma_i \longmapsto \begin{bmatrix} I & & & \\ & 0 & t & \\ & 1 & 0 & \\ & & & I \end{bmatrix}$$

n-dim'l irrep.

f.d. Quotients of $K[B_n]$ algebras

Eg: Hecke

$$\boxed{\mathcal{H}_n(t) = \mathbb{Q}(t)\,[B_n] \Big/ \langle (\sigma_i - 1)(\sigma_i + t) \rangle}$$

$\dim(\mathcal{H}_n) = n!$

t generic.

Rep of $\mathcal{H}_n \rightsquigarrow B_n$ reps.

The Yang-Baxter Eqn

$R \in \mathrm{Aut}(V \otimes V)$: $R_1 = R \otimes I_V$, $R_2 = I_V \otimes R$

(YBE) $\quad R_1 R_2 R_1 = R_2 R_1 R_2$

$\sigma_i \longmapsto R_i = I_V^{\otimes(i-1)} \otimes R \otimes I_V^{\otimes(n-i-1)}$ is a B_n rep.

on $V^{\otimes n}$. (R, V) a Braided Vector Space.

$$\boxed{\text{Classified up to } \dim(V) = N \leq 2 \quad \text{Hietarinta 1992}}$$

Reps of $\mathcal{LB}_n$.

1. $\boxed{\text{Lift}}$ reps of B_n

Buran, "Standard" etc. ✓

2. (R, S, V) Loop braided V.S.

$\sigma_i \longmapsto R_i$ $\quad$ $s_i \longmapsto S_i = I \otimes \cdots \otimes S \otimes \cdots \otimes I$

sometimes ✓, sometimes ✗ ...

3. f.d. Quotients of $K[\mathcal{LB}_n]$ K a field.

Finite Dimensional $K[\mathbb{L}\mathbb{B}_n]$ quotients?

w/ Damiani & Martin

$\mathbb{Q}(q)\mathbb{L}\mathbb{B}_n / \langle (\sigma_i - 1)(\sigma_i + q) \rangle$ not f.d.

But... Thm: (DMR)

$$\mathbb{C}(q)\mathbb{L}\mathbb{B}_n / \langle (\sigma_i - 1)(\sigma_i + q), (\sigma_i - 1)(s_i + 1), (s_i - 1)(\sigma_i + q) \rangle$$

is f.d.!

Call it $\mathbb{L}\mathcal{H}_n(q)$: Loop Hecke algebra.

Admits a "local" rep.:

$$R = \begin{bmatrix} 1 & & & \\ & 1-q & q & \\ & 1 & 0 & \\ & & & -q \end{bmatrix} \qquad S = R\big|_{q=1}$$

Look!

$\sigma_i \longmapsto R_i$, $s_i \longmapsto S_i$ gives a rep of $\mathcal{TH}_n$ on $(\mathbb{C}^2)^{\otimes n}$.

$\boxed{\text{Conj. (DMR) This rep is faithful! } (\checkmark \text{ for } n \leq 7) \quad (\text{for } \mathcal{TH}_n, \text{ not } \mathcal{TB}_n)}$

Image: $\mathbb{C}(q)\langle R_i, S_i \rangle_{i=1}^{n-1}$ for $q^2 \neq 1$

has dim: $\frac{1}{2}\binom{2n}{n}$

Question: What is special about $\left[\begin{array}{c|cc|c} 1 & & & \\ \hline & 1-q & q & \\ & 1 & 0 & \\ \hline & & & -q \end{array}\right]$?

From: $U_q\mathfrak{sl}_{(1|1)}$

One Answer: charge conserving!

Charge Conserving YBOs

$\underline{V}$, $B = \{e_1, \ldots, e_N\}$. Denote: $e_i \otimes e_j =: |ij\rangle$

& $V^{ij} = \mathbb{C}\{|ij\rangle, |ji\rangle\}$. $T \in \mathrm{End}(V^{\otimes 2})$

is charge conserving if $T(V^{ij}) \subset V^{ij}$

for all i, j.

$V = \mathbb{C}^2$:

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & x & c & 0 \\ 0 & b & y & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

$\dim(V) = 3$

form:

$$\left(\begin{array}{ccc|ccc|ccc}
a_{11} & & & & & & & & \\
 & a_{12} & & b_{12} & & & & & \\
 & & a_{13} & & & & b_{13} & & \\
\hline
 & c_{12} & & d_{12} & & & & & \\
 & & & & a_{22} & & & & \\
 & & & & & a_{23} & & b_{23} & \\
\hline
 & & c_{13} & & & & d_{13} & & \\
 & & & & & c_{23} & & d_{23} & \\
 & & & & & & & & a_{33}
\end{array}\right)$$

General form: $T\big|_{V^{ij}} = A(i,j)$ $\quad T\big|_{V^{ii}} = a_i$

$i < j$

$$A(i,j) = \begin{bmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{bmatrix}$$

Encode as:

$$\boxed{(a_1, \dots, a_N, A(1,2), \dots, A(N-1,N))}$$

$\binom{N}{2} \cdot 4 + N$ scalars to determine...

Problem: Classify charge conserving Yang-Baxter operators up to symmetries

Lemma: If $T \in \mathrm{End}(V^{\otimes 2})$ is a CCYBO

(1) so is: $P_\gamma \otimes P_\gamma \, T \, (P_\gamma \otimes P_\gamma)^{-1}$ $\quad P_\gamma(e_i) = e_{\gamma(i)}$ $\gamma \in S_n$

(2) so is: $X T X^{-1}$ $\quad X$ diagonal "X-equivalence"

Remarks:

(1) General $Q \otimes Q$ conj. destroys CC

(2) X-equiv. is not enjoyed by general YBOs.

(3) Goal: classify up to (1) & (2)

$N=2$ Solutions

Up to Symmetries...

$$\begin{bmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \alpha \end{bmatrix}$$

$$\left[\begin{array}{c|cc|c} \alpha & & & \\ \hline & \alpha+\beta & -\beta & \\ & \alpha & 0 & \\ \hline & & & \alpha \mid \beta \end{array}\right]$$

$$\left[\begin{array}{c|cc|c} \alpha & & & \\ \hline & 0 & \mu & \\ & \mu & 0 & \\ \hline & & & \beta \end{array}\right]$$

pick one!

$\alpha+\beta \neq 0$

$\alpha, \beta, \mu \neq 0.$

Observe: if R is a c.c. YBO, $R|_{V^S}$ is too:

$S \subseteq \{1, \dots, N\}$

$V^S = \mathbb{C}\{|ij\rangle : i,j \in S\}.$

Proof: Calculate from general form:

$$R|_{V^{ij}} = \begin{bmatrix} a_i & 0 & 0 & 0 \\ 0 & a_{ij} & b_{ij} & 0 \\ 0 & c_{ij} & d_{ij} & 0 \\ 0 & 0 & 0 & a_j \end{bmatrix}$$

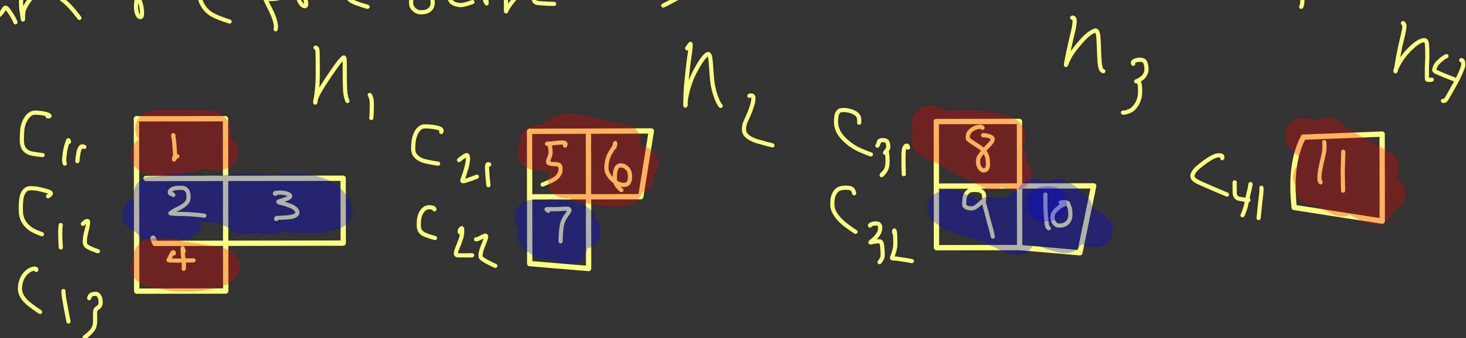
A combinatorial Parametrization

$\underline{N} := \{1, 2, \dots, N\}$ individuals.

1. Partition $\underline{N}$ into k "nations" $n_1, \dots, n_k$ respect order: $1 \in n_1$ & $N \in n_k$ etc.
2. Break nations into "counties" $n_i = \bigsqcup_j c_{ij}$ again respect order.
3. Color counties or s.t. 1st counties c_{1j} are

Can represent as bi-colored "composition tableaux".

Ex:



Now define $(a_1, \dots, a_N, A(1,2), \dots, A(N-1,N))$ as follows.

$$i \in C_{sx} : \quad a_i = \begin{cases} \alpha_s & C_{sx} \\ \beta_s & C_{sx} \end{cases} \qquad \begin{matrix} \alpha_s \neq 0 \\ \beta_s \neq 0 \end{matrix} \qquad \alpha_s + \beta_s \neq 0.$$

$i<j$	$A(i,j)$	params.
$i \in n_s$, $j \in n_t$, $s \neq t$	$\begin{pmatrix} 0 & \mu_{st} \\ \mu_{st} & 0 \end{pmatrix}$	$\mu_{st} \neq 0$ for each pair of nations
$i \in C_{sx}$, $j \in C_{sy}$, $x \neq y$	$\begin{pmatrix} \alpha_s + \beta_s & -\beta_s \\ \alpha_s & 0 \end{pmatrix}$	$\alpha_s \neq 0 \neq \beta_s$, $\alpha_s + \beta_s \neq 0$ for each nation
$i, j \in C_{sx}$	$\begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}$	$\gamma = a_i = \begin{cases} \alpha_s & C_{sx} \\ \beta_s & C_{sx} \end{cases}$

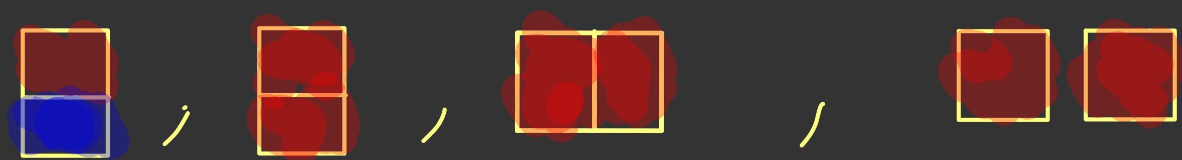
Thm: (1) This a sol'n.
(MR)

(2) Up to symmetries, gives all sol'ns.

(3) Up to changes of variables & permuting nations form a transversal.

Enumeration: orbits of sol'n varieties in bij. with

$\boxed{\text{Multisets of bicolored comp. tableaux}}$

N=2: 

Euler Transform of 3^{N-1}: 1, 4, 13, 46, 154, 533, ...

Question: When do cc YBOs lift to $\mathcal{LB}_n$?

Thm: (MRT) Lift if & only if bi-colored composition tableaux has at most 2 rows & distinct rows are distinct colors.

Thm: (MRT) classif. 4

$$\begin{cases} S = (\varepsilon_1, \ldots, \varepsilon_N, \ldots B(i,j) \ldots) \ \& \\ R = (a_1, \ldots, a_N, \ldots A(i,j) \ldots) \end{cases}$$

c.c. (S,R) Loop Braided Vector Space

As before, $A(i,j)$ & $B(i,j)$ depend on residency of i,j.

Lemma: Same symmetries hold: $P_\gamma \otimes P_\gamma$, $\gamma \in \mathcal{S}_n$

$\mathbb{X}$ diag.

Combinatorial Parametrization of orbits of Solh varieties

$\boxed{\text{Pairs}}$ of Multisets (M_1, M_2) where

M_i consists of 2-row comp. tableaux

A wrinkle: $\boxed{\text{Pairs}}$ $\longleftrightarrow$ signs...

Ex:

$$\Big(\underbrace{\begin{ytableau}1\\2&3\end{ytableau}\quad\begin{ytableau}4\\5\end{ytableau}\quad\begin{ytableau}6\end{ytableau}}_{+}\;;\;\underbrace{\begin{ytableau}7&8\end{ytableau}\quad\begin{ytableau}9\\10\end{ytableau}\quad\begin{ytableau}11\\12\end{ytableau}}_{-}\Big)\qquad N=12.$$

$\varepsilon_1 = 1$

$\varepsilon_2 = -1$

$\varepsilon_9 = -1$

$\varepsilon_{10} = 1$

$B(7,8) = -\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$A(7,8) = -\begin{pmatrix}\alpha_4&0\\0&\alpha_4\end{pmatrix}$

Enumeration: Orbits of Solin varieties

$\updownarrow$ 1-1 "stable corner

Pairs of Plane partitions Box stacking"

$\pi_{i,j}$: $\pi_{i,j+1} \leq \pi_{i,j} \geq \pi_{i+1,j}$ $\sum_{i,j} \pi_{i,j} = N$

Ex: $\begin{matrix} 4 & 2 & 2 \\ 3 & 2 & \\ 1 & 1 & \end{matrix} \in PL(15)$

Generating fnc. $\prod_{n\geq 1} \frac{1}{(1-x^n)^{2n}}$: 1, 2, 7, 18, 47, 110, 258, ...

See paper for details...

Consider each pair of individuals $i < j$.

1. If $i \in n_s$ and $j \in n_t$ with $s \neq t$ then $A(i,j) = \begin{pmatrix} 0 & \mu_{s,t}/C_{s,t} \\ \mu_{s,t}C_{s,t} & 0 \end{pmatrix}$, and $B(i,j) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
2. If i and j are in the same nation n_t but different counties $s_{t,x}$ and $s_{t,y}$ (note $x < y$ by construction), then $A(i,j) = \begin{pmatrix} \alpha_t + \beta_t & \alpha_t \\ -\beta_t & 0 \end{pmatrix}$ and $B(i,j) = \operatorname{sgn}(n_t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
3. If i and j are both in the first, (respectively second), county in n_t then $A(i,j) = \operatorname{sgn}(n_t) \begin{pmatrix} \alpha_t & 0 \\ 0 & \alpha_t \end{pmatrix}$, (respectively $A(i,j) = \begin{pmatrix} \beta_t & 0 \\ 0 & \beta_t \end{pmatrix}$) and $B(i,j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Future:

- prove conj. on $\mathcal{JH}_n$.
- new algebras from (R,s)?
- categorification? Symmetries of ?
- Loop BMW...?
- Invariants (of 4-mflds?)

Thank
you!