Stable Khovanov homology of torus links and volume[1]

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Moduli and Friends
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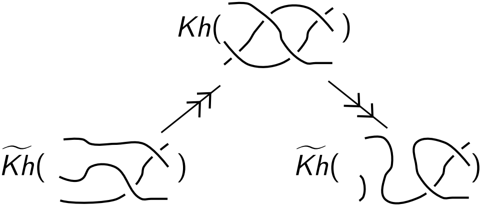
Stable Khovanov homology of infinite torus links

Theorem (Stosic 2007)

Let $T_{n,k}$ be the torus link on n strands. For $2 \leq n < k$, and $i < n + k - 3$,

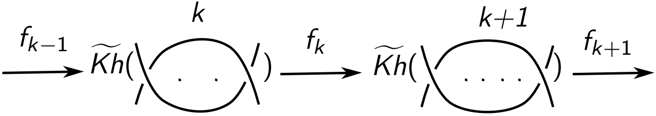
$$Kh_{i,j}(T_{n,k}) \cong Kh_{i,j}(T_{n,k+1}) \qquad \textit{(Stosic 2007)}$$



Theorem (Rozansky 2014)

The stable homology $\mathcal{H}_\infty$ of infinite torus braids $\widehat{\beta_{n,k}} = (\sigma_1\sigma_2\cdots\sigma_{n-1})^k$ is the direct limit of the directed system $\left\{\widetilde{Kh}_{*,*}(T_{n,k}), f_k\right\}$ of (shifted) Khovanov homologies.

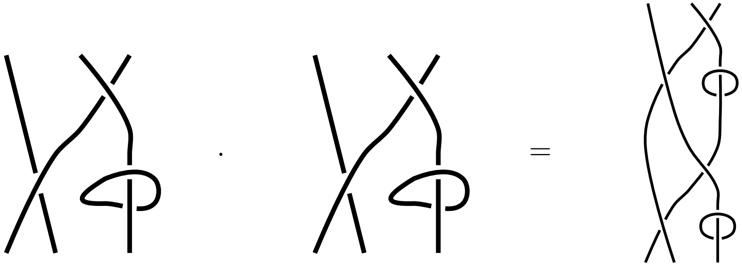
$$\mathcal{H}_\infty(\beta_{n,k}) := \varinjlim \widetilde{Kh}_{*,*}(\beta_{n,k}) \qquad \textit{(Rozansky 2014)}$$



Furthermore, Rozansky showed $\mathcal{H}_\infty$ gives a categorification of the Jones-Wenzl projector **P**, an important object in quantum topology.

The Temperley-Lieb algebra

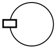
$TL_n :=$ $\mathbb{C}$-vector space of formal linear sums of link/tangle diagrams in a disk D with n marked points above and below, modded out by the Kauffman bracket skein relations.



Multiplication $\cdot$ is done by stacking.

The Jones-Wenzl projector

Let A be an indeterminate. Then there is a unique element $\boxminus_n$ such that

- $\boxminus_n \cdot \underbrace{|\asymp|^i_n}_{e_i} = 0 = |\asymp|^i_n \cdot \boxminus_n$
- $\boxminus_n - |_n$ belongs to the algebra generated by $\{|\asymp|^i_n\}_{i=1}^{n-1}$
- $\boxminus_n \cdot \boxminus_n = \boxminus_n$
-  $_n = (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$

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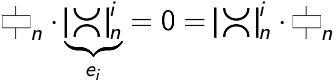
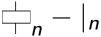
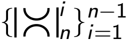
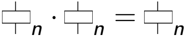
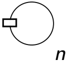
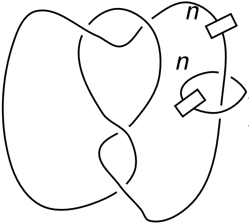
- 
-  belongs to the algebra generated by 
- 
-  $= (-1)^n \frac{A^{2(n+1)} - A^{-2(n+1)}}{A^2 - A^{-2}}$

The complex $\mathbf{P}_n$ is the categorification of  (Cooper-Krushkal 2012), (Rozansky 2014).

The colored Jones polynomial

The colored Jones polynomial of a link L is obtained by cabling an nth Jones-Wenzl projector in each component.

$$J_L(q;n) := (-1^{n-1} q^{n^2-1})^{w(D)} \left\langle \quad \right.$$



$$\left. \quad \right\rangle \Big|_{A=q^{-1/4}}$$

The volume conjecture

Conjecture (Kashaev 1997, Murakami and Murakami 2001)

Let K be a hyperbolic knot, then

$$2\pi \lim_{n\to\infty} \frac{\log |\widehat{J_K}(e^{\frac{\pi i}{2n}}; n)|}{n} = vol(S^3 \setminus K).$$

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The conjecture is known for

- The figure 8 knot (Ekholm).
- Whitehead chains (van der Veen 2008).
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For links it is known for Whitehead doubles (Zheng 2007) and a version of the volume conjecture is known for augmented Knotted Trivalent Graphs (van der Veen 2009).

Relationship to hyperbolic geometry

Let M be a manifold with c cusps, and let $M_{(k_1,\dots,k_c)}$ for $k_i \in \mathbf{Q} \cup \infty$ denote the result of doing k_i-Dehn surgery on the ith cusp of M.

Theorem (Thurston hyperbolic Dehn surgery)

For all $\vec{k} = (k_1, \dots, k_c)$ *such that* $k_i >> 0$,

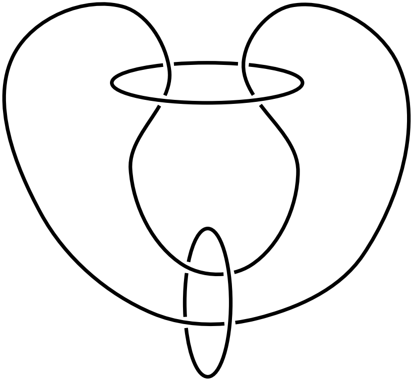
$$\lim_{\vec{k}\to\infty} M_{\vec{k}} = M,$$

so

$$\lim_{\vec{k}\to\infty} vol(M_{\vec{k}}) = vol(M)$$

and

$$Vol(M_{\vec{k}}) < Vol(M) \qquad (\vec{k} \neq \infty).$$



Surgery on the crossing circles produces the figure 8 knot

Question

(How) does this reflect in the quantum picture?

Theorem (Champanekar-Kofman 2005)

The coefficient vector of the Jones polynomial and the colored Jones polynomial decomposes into fixed blocks that separate as the number of twisting increases.

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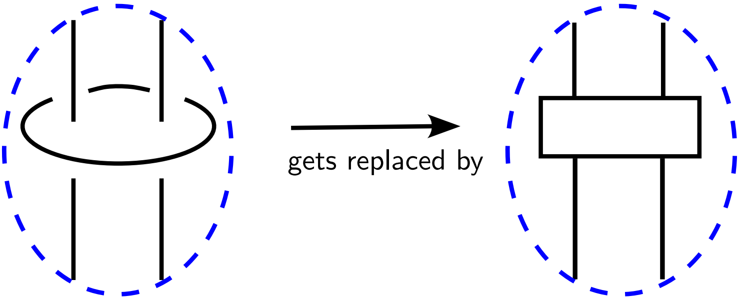
[1 -1 2 -1 2 -1 1] 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 [2 -8 16 -23 20 -12 0 7 -7 4 -1] 0 0 0 0 0 0
0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
[1 -5 15 -29 40 -42 33 -19 8 -2]

Src: Champanekar-Kofman

A quantum Dehn surgery-type theorem

For a fully augmented link L_∞ with c twist regions, the skein element L_J^n is obtained by replacing every 2-tangle encircled by a crossing circle by a $2n$ Jones-Wenzl projector.



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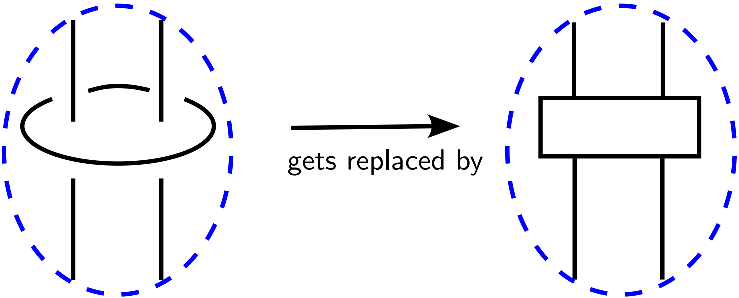

Theorem (L.)

Let $k_i >> 0$ and fix an integer $n \geq 1$, the homology groups $\{KH(L_{k_1,\ldots,k_c}; n)\}$ form a directed system, and its direct limit as $k \to 0$ is given by $KH(L_J^n)$.

- The proof iterates Rozansky's construction,
- combined with Willis-Islambouli's (2018) result on being able to recover the Jones-Wenzl idempotent from any infinite positive braid.

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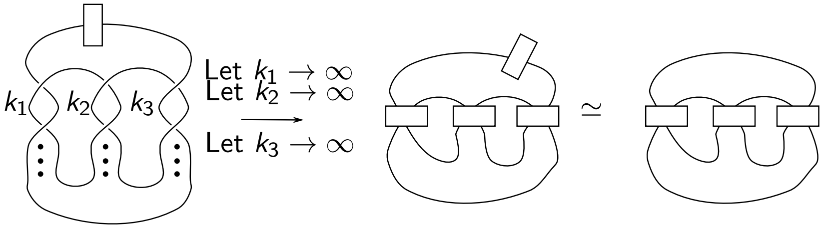
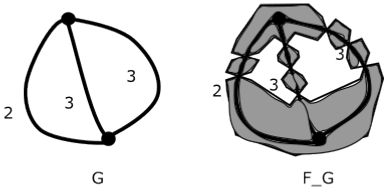


Connection with volume

Let G be a $\mathbb{Z} \setminus \{0\}$-weighted planar graph, and let F_G be the surface obtained by replacing each weighted edge by a twisted band, and each vertex with a disk. Let $L = \partial(F_G)$.



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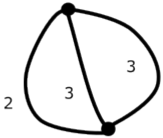

G

F_G

Theorem (L.)

Let T be the number of triangular moves in a sequence S of moves to obtain G, then

$$2\pi(T+1)v_8 \leq 2\pi \lim_{n\to\infty} \frac{\log |\widehat{J_G}(e^{\frac{\pi i}{2n}}; n)|}{n},$$

with equality if there are no fusion moves in S.

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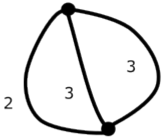

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The geometry of fully augmented links

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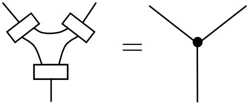
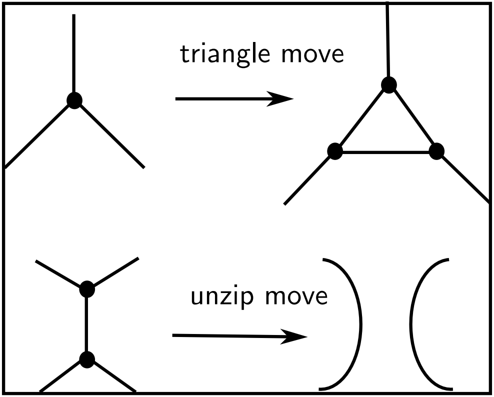
Proposition

Let L be a hyperbolic fully augmented link with c augmentation circles. Then its volume is at least $2v_8(c-1)$, *where* $v_8 = 3.66386\cdots$ *is the volume of a regular ideal octahedron. Moreover, the volume is exactly* $2v_8(c-1)$ *if and only if* $S^3 \setminus L$ *decomposes into regular ideal octahedra.*

The colored Jones polynomial of KTG's





Every L_J^n can be obtained from the n-colored unknot through a sequence of triangle and unzip moves.

The $(SO(3))$ volume conjecture for KTG's

Conjecture (van der Veen 2009)

The following form of the volume conjecture holds for all knotted trivalent graphs

$$\lim_{n\to\infty} \frac{2\pi}{n} \log |\widehat{J_\Gamma}(e^{\frac{\pi i}{2n}}; n)| = vol(\Gamma),$$

where n runs over the even numbers only.

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Theorem (van der Veen 2009)

Let S be a sequence of KTG moves, there exists an $n \in \mathbb{N}$ such that all n-augmented KTG's satisfy the $SO(3)$ volume conjecture, but not the original volume conjecture.

Theorem (van der Veen 2009)

In particular, let T be the number of triangle moves in S and let r be the number of crossing circles of Γ. Let θ be the number of half twists counted with sign. The colored Jones invariant of Γ satisfies

$$J_n(\Gamma; e^{\frac{\pi i}{2n}}) = \begin{cases} n^r \mathbf{sixj}_n^{T+1} \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd} \end{cases}$$

Therefore, by (Constantino 2007),

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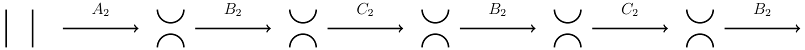
Obs: If S does not contain any fusion move, then L_J^n is an augmented KTG, and so the Theorem applies.

Potential application to the volume conjecture (in progress)

Working in reverse from Rozansky's result, $Kh(T(2,k))$-tangle can be recovered from a truncation of $\mathbf{P}_2$.



(Manion 2018)

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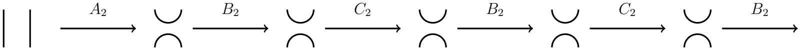

(Manion 2018) Similarly, $Kh(T(n,k))$ can be recovered from a truncation of $\mathbf{P}_n$. Thus we can describe the formal difference

$$\widehat{J}_{L_J}(e^{\frac{\pi i}{2n}}; n) - \widehat{J}_{L_k}(e^{\frac{\pi i}{2n}}; n)$$

as consisting of periodic complexes.

Thank you for listening