BUCHAREST NOTES
26.06.2023

Freeness of conic-line arrangements.

based on a joint projects with Alex Dimca

Main Motivation: to understand arrangements of conics & lines in $\mathbb{P}^2_{\mathbb{C}}$ that are free / nearly-free.

Origins: Terao's freeness conjecture for line arrangements.

Let $\mathcal{L} = \{\ell_1, \ldots, \ell_d\} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of lines in $\mathbb{P}^2_{\mathbb{C}}$.

We define the freeness of $\mathcal{L}$ via result due to Du Plessis & Wall.

Denote by $Q \in \overset{S}{\overset{\shortparallel}{\mathbb{C}[x,y,z]}}$ the defining equation of $\mathcal{L}$ (up to constant).

We define $\operatorname{mdr}(Q) = \min\limits_{\substack{r \geq 0 \\ \text{integer}}} \{ (a,b,c) \in S_r^{\oplus 3} : a \cdot \partial_x Q + b\, \partial_y Q + c\, \partial_z Q = 0 \}$

= the minimal degree of the Jacobian relations among the partials.

Def / Thm: Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of lines of degree $d \geq 2$. Assume that $d_1 := \operatorname{mdr}(Q) \leq \frac{d-1}{2}$. Then $\mathcal{L}$ is free iff

$$d_1^2 - d_1(d-1) + (d-1)^2 = \underbrace{\sum_{p \in \operatorname{Sing}(\mathcal{L})} (\operatorname{mult}_p(\mathcal{L}) - 1)^2}_{\tau(\mathcal{L})}.$$

the Total Tjurina number of $\mathcal{L}$.

Ex: $Q(x,y,z) = xyz$

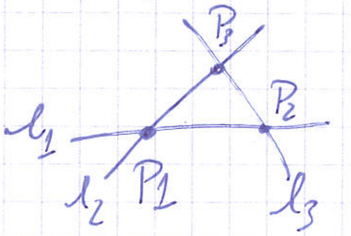
$d_1 = 1$ since we have $x \cdot \partial_x Q - y \cdot \partial_y Q = 0$.

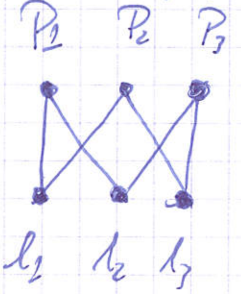
We have $3 = 1 - 1 \cdot 2 + 4 = \sum_{p \in \operatorname{Sing}(\mathcal{L})} (\operatorname{mult}_p - 1)^2 = 1 \cdot 3$, so $\mathcal{L}$ is free.

Df: Let $\mathcal{L}$ be an arrangement of lines in $\mathbb{P}^2_{\mathbb{K}}$.

The Levi graph $\mathcal{G}(\mathcal{L})$ associated to $\mathcal{L}$ is a bipartite graph with vertices $\{x_1, \dots, x_d\}$ corresponding to lines, $\{y_1, \dots, y_s\}$ corresponding to points (singular), and x_i is incident with y_j if the corresponding point is incident with the associated line.

For $Q(x,y,z) = xyz$ the corresponding Levi graph looks as follows.





Terao's conjecture:

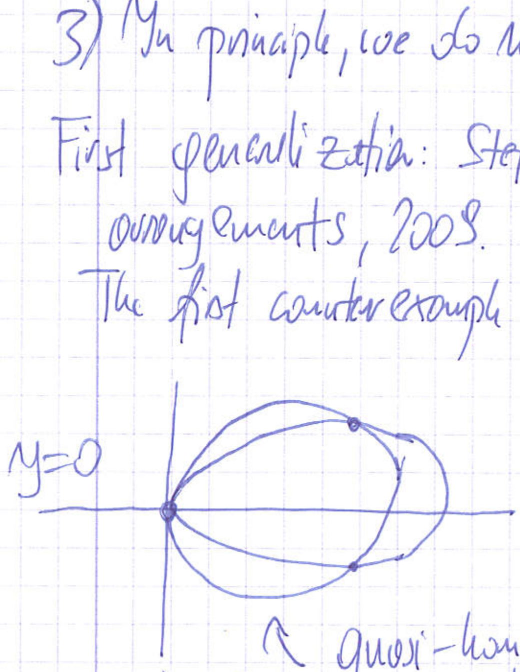
Let $\mathcal{L}_1, \mathcal{L}_2$ be two line arrangements of lines, $\mathcal{G}(\mathcal{L}_1)$, $\mathcal{G}(\mathcal{L}_2)$ the corresponding Levi graphs. Assume that $\mathcal{L}_1$ is free & $\mathcal{G}(\mathcal{L}_1)$, $\mathcal{G}(\mathcal{L}_2)$ are isomorphic, then $\mathcal{L}_2$ has to be free.

What do we know?

1) It holds with up to 14 lines (Kühne, Barakat ~2023).
2) It holds for line arrangements with up to 13 lines (Dimca, Ibadula, Macinic, 2019)
3) In principle, we do not know much about ~~this~~ problem.

First generalization: Stefan Tohaneanu & Hal Schenck, Conic-line arrangements, 2009.

The first counterexample to a naive Terao's freeness problem.



↖ quasi-homogeneous singularity

If we replace by $y=0$ by $x-13y=0$, then combinatorially these are the same guys, but the singularity is not quasi-homogeneous!

This is not quite right since we forget about the topological type of singularities...

We use a slightly different approach via Vallès & Marchesi definition.

Setting: Let $\mathcal{CL} = \{l_1, \dots, l_d, C_1, \dots, C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $d \geq 0$ lines & $k \geq 0$ smooth conics.

By the <u>weak combinatorics</u> we mean: $(d, k; \#S_1, \dots, \#S_t)$, where S_i denotes the singular point of a given topological type.

To give you some feeling, for line arrangements we have

$(d; t_2, \dots, t_d)$, where t_i denotes the number of i-fold intersection points.

First main goal: suggested by Hal Schenck;
Understand free conic arrangements.

<u>Our setting</u>: we allow to have
k smooth conics
n_2 nodes
n_3 ordinary triple points
n_4 ordinary quadruple points
t_2 tacnodes.

We can count $4\binom{k}{2} = n_2 + 2 \cdot t_2 + 3 n_3 + 6 n_4$.

all singularities here are quasi-homogeneous $\Leftrightarrow$ $\tau_p = \mu_p$.

In principle, we can have many possibilities

Thm: (—, PAMS 2023). There is no free arrangement of $k \geq 2$ smooth conics with node, tacnodes, ordinary triple & quadruple points.

It is really bad somehow.

Remark: There are free arrangements of conics with quasi-homogeneous singularities: Two conics with A_7 singularity.

Persson's triconical arrangement with $2A_7$, $1A_3$, $2A_1$.

However:

Thm (, 2023).

Let $\mathcal{C}_K$ be an arrangement of $k \geq 2$ smooth conics admitting nodes, tacnodes, ordinary triple, A_5 & A_7 singularities. Assume that $\mathcal{C}_k$ is free.

Then $k \in \{2,3,4\}$, possibly except $k=4$.

In order to exclude $K=4$ we need to decide whether 10 weak combinatorics cannot be constructed geometrically by 4 smooth conics.

This problem is difficult!

Conics & lines in the plane (joint work with Alex Dimca).

We want to understand the freeness of conic-line arrangements with quasi-homogeneous singularities.

Here is our setting.

$\mathcal{CL} = \{\ell_1, \dots, \ell_d, C_1, \dots, C_k\} \subseteq \mathbb{P}^2_{\mathbb{C}}$ $d \geq 1$ lines & $k \geq 1$ smooth conics.

We assume that they admit only nodes n_2, tacnodes t, ordinary triple points n_3.

All results published in JOAC 2022.

Thm: If $2k+d \geq 12$, then

$$20k + n_2 + \frac{3}{4} n_3 \geq d + 4t$$

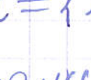
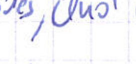
Prop: Let $C : f=0$ be a reduced curve of degree m in $\mathbb{P}^2_{\mathbb{C}}$ having only nodes, tacnodes, and ordinary triple points.

Then $\mathrm{mdr}(f) \geq \frac{2}{3} m - 2$ &

if C is free, then $m \leq 9$.

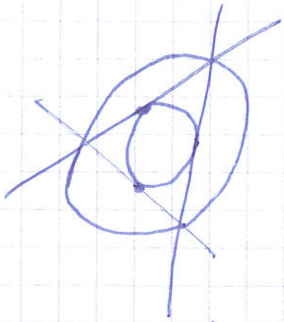
Main Result $\mathcal{CL} = \{\ell_1, \dots, \ell_d, C_1, \dots, C_k\} \subset \mathbb{P}^2_{\mathbb{C}}$ an arrangement of $d \geq 1$ lines & $k \geq 1$ smooth conics with nodes, tacnodes, and ordinary triple points.

Then $\mathcal{CL}$ is free iff one of the following cases occur:

- $d=k=1$,  $t=1$
- $d=2$, $k=1$, $n_2=1$, $t=2$, $n_3=0$ 
- a) $d=3$, $k=1$  $t=3$, $n_2=3$, $n_3=0$

  b) $d=3$, $k=1$  $t=n_2=0$, $n_3=3$

•) $d=3$, $k=2$
$n_3=3$, $t=5$



Corollary: Since all arrangements above are unique up to projective equivalence, Numerical Terao's Conjecture holds for conic-line arrangements with nodes, tacnodes, ordinary triple points.
Using our methods we can also show that Numerical Terao's Conjecture holds for line arrangements with double & triple intersection points!

$\boxtimes$

Concluding result:

Thm: (–, 2023) Let $\mathcal{C}=\{C_1,\dots,C_k\}\subset\mathbb{P}^2_{\mathbb{C}}$ be an arrangement of smooth curves, each of degree $d\geq 3$. Assume that $\mathcal{C}$ admits only ordinary singularities of multiplicity <5 (= quasi-homogeneous). Then $\mathcal{C}$ is never free.