# Multivariable knot polynomials from Nichols algebras


Rinat Kashaev

University of Geneva


November 6, 2023

---

Ongoing joint work with Stavros Garoufalidis

# Motivation

The powerful standard approach to quantum knot invariants is through Drinfeld's quantum double construction:

1. Input: $H$ (Hopf algebra with invertible antipode);
2. $D(H)$ the quantum double (Hopf algebra with universal R-matrix);
3. Finite dimensional representation $h\colon D(H) \to \mathrm{End}(V)$;
4. Evaluation of the universal R-matrix at $h$;
5. A ribbon element or enhancement;
6. Output: knot/link invariants through the Reshetikhin–Turaev functor.

Drawback: Commutative and co-commutative Hopf algebras do not produce non-trivial invariants.

**Question:** *Is there a shorter and/or simpler approach?*

Partial answer: The three steps 2–4 can be replaced by the construction of finite-dimensional Yetter–Drinfel'd modules and their braided structure.

# Today's message

A more direct approach towards quantum knot invariants:

1. Input: H (*braided* Hopf algebra with invertible antipode endowed with an *automorphism*);
2. $Y$ (appropriately adapted finite-dimensional Yetter–Drinfel'd module);
3. Output: End($Y$)-valued invariant of long knots.

## Remark

Commutative and co-commutative Hopf algebras can produce non-trivial invariants if they admit non-trivial automorphisms.

## Example

Input: two-dimensional super Hopf algebra $H = \mathbb{F}[x]/(x^2)$ with odd primitive generator $x$ endowed with the automorphism $\phi_t \colon H \to H$, $\phi_t x = tx$, $t \in \mathbb{F}_{\neq 0}$.
Output: $\Delta_K(t)\,\mathrm{id}_H$ with the (canonically normalized) Alexander polynomial $\Delta_K(t)$.

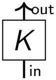
# Outline

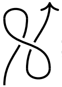
- Quantum invariants of long knots
- Functorial Hopf objects
- Yetter–Drinfel'd functorial objects
- Nichols algebras
- Analysis of rank 1
- An example in rank 2

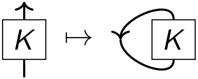
# Quantum invariants of long knots

## A variation of the Reshetikhin–Turaev construction

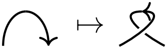
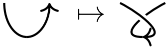
An (oriented) long knot diagram $K$ is an oriented knot diagram in $\mathbb{R}^2$ with two open ends called "in" and "out":

$K =$ 


Examples: $K =$  ; $K =$ 

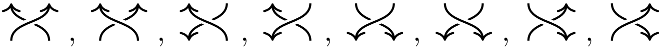
Relation to closed diagrams: 


Normalization: diagram $\dot{K}$ obtained from $K$ by the replacements

 and 

$K$ is called normal if $K = \dot{K}$.

Building blocks: four types of segments  and eight types of crossings



## Rigid R-matrices

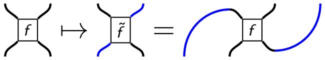
An R-matrix over a vector space $V$: $r \in \mathrm{Aut}(V \otimes V)$ satisfies the quantum Yang–Baxter relation

$$r_1 r_2 r_1 = r_2 r_1 r_2, \quad r_1 := r \otimes \mathrm{id}_V, \ \ r_2 := \mathrm{id}_V \otimes r.$$

Assume $\dim V < \infty$, $B \subset V$ (a basis), $\{b^*\}_{b \in B} \subset V^*$ (the dual basis).

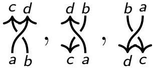
$f \in \mathrm{End}(V \otimes V) \mapsto$ partial transpose $\tilde{f} \colon V^* \otimes V \to V \otimes V^*$

$$\tilde{f}(a^* \otimes b) = \sum_{c,d \in B} \langle a^* \otimes c^*, f(b \otimes d) \rangle c \otimes d^*,$$




R-matrix $r$ is called rigid if $\widetilde{r^{\pm 1}}$ are invertible.

Given $r$ (rigid R-matrix over $V$, $\dim V < \infty$), $B \subset V$ (basis), $K$ (normal long knot diagram), $E_K$ (edges of $K$), $C_K$ (crossings of $K$), $s\colon E_K \to B$ (state of $K$).
Define the weight of $K$ (in state $s$) $w_s(K) = \prod_{c \in C_K} w_s(c)$,



$$\xmapsto{w_s} \langle c^* \otimes d^*, r(a \otimes b)\rangle,$$

$$\xmapsto{w_s} \left\langle a \otimes c^*, \left(\widetilde{r^{-1}}\right)^{-1}(b \otimes d^*)\right\rangle$$

and likewise for negative crossings with the replacements $r \leftrightarrow r^{-1}$.

### Theorem

*Let a normal long knot diagram $K$ have an equal number of negative and positive crossings. Then, the linear map*

$$J_r(K)\colon V \to V, \quad J_r(K)a = \sum_{s \in B^{E_K},\ s_{in} = a} w_s(K) s_{out}$$

*is a knot invariant.*

### Remark

This construction can be extended to the context of arbitrary monoidal categories with duality.

# Hopf functorial objects

A functor $F\colon \mathcal{D} \to \mathcal{C}$ is called *functorial object* (or *f-object*) if it is considered as an object of the functorial category $\mathcal{C}^{\mathcal{D}}$ with natural transformations as morphisms.
Consider the group $\mathbb{Z}$ as a category/groupoid with one object. Then, the objects of the functorial category $\mathcal{C}^{\mathbb{Z}}$ are the pairs $(A, \phi)$ where $A \in \mathrm{Ob}(\mathcal{C})$ and $\phi \in \mathrm{Aut}(A)$, and a morphism from $(A, \phi)$ to $(B, \psi)$ is a morphism $f\colon A \to B$ of $\mathcal{C}$ such that $\psi f = f\phi$. If $\mathcal{C}$ is braided monoidal then so is $\mathcal{C}^{\mathbb{Z}}$.

## Definition

Let $\mathcal{C}$ be a braided monoidal category. A *Hopf functorial object* is a Hopf object $H$ in the braided monoidal category $\mathcal{C}^{\mathbb{Z}}$ with product $\nabla\colon H \otimes H \to H$, unit $\eta\colon \mathbb{I} \to H$, coproduct $\Delta\colon H \to H \otimes H$, counit $\epsilon\colon H \to \mathbb{I}$, and antipode $S\colon H \to H$.

We will always assume that $S$ is an invertible (functorial) morphism.

# Yetter–Drinfel'd functorial objects

A *left Yetter–Drinfel'd functorial object* over a Hopf functorial object $H \in \mathrm{Ob}(\mathcal{C}^{\mathbb{Z}})$ is a triple $(Y, \lambda, \delta)$ where $Y \in \mathrm{Ob}(\mathcal{C}^{\mathbb{Z}})$, and morphisms $\lambda\colon H \otimes Y \to Y$, $\delta\colon Y \to H \otimes Y$ are such that $(Y, \lambda)$ is a left $H$-module f-object, $(Y, \delta)$ is a left $H$-comodule f-object, and

$$\begin{aligned}(\nabla \otimes \mathrm{id}_Y)(\mathrm{id}_H \otimes \tau_{Y,H})(\delta\lambda \otimes \phi_H)(\mathrm{id}_H \otimes \tau_{H,Y})(\Delta \otimes \mathrm{id}_Y)& \\ = (\nabla \otimes \lambda)(\mathrm{id}_H \otimes \tau_{H,H} \otimes \mathrm{id}_Y)(\Delta \otimes \delta)&\end{aligned}$$

where $\phi_H\colon H \to H$ is the functorial isomorphism that at the unique object $*$ of $\mathbb{Z}$ evaluates as

$$(\phi_H)_* = H(1)\colon H(*) \to H(*).$$

A *right Yetter–Drinfel'd functorial object* over a Hopf functorial object $H \in \mathrm{Ob}(\mathcal{C}^{\mathbb{Z}})$ is a triple $(Y, \lambda, \delta)$ where $Y \in \mathrm{Ob}(\mathcal{C}^{\mathbb{Z}})$ and morphisms $\lambda\colon Y \otimes H \to Y$, $\delta\colon Y \to Y \otimes H$ are such that $(Y, \lambda)$ is a right $H$-module f-object, $(Y, \delta)$ is a right $H$-comodule f-object, and

$$\begin{aligned}(\mathrm{id}_Y \otimes \nabla)(\tau_{H,Y} \otimes \mathrm{id}_H)(\phi_H \otimes \delta\lambda)(\tau_{Y,H} \otimes \mathrm{id}_H)(\mathrm{id}_Y \otimes \Delta)& \\ = (\lambda \otimes \nabla)(\mathrm{id}_Y \otimes \tau_{H,H} \otimes \mathrm{id}_H)(\delta \otimes \Delta)&.\end{aligned}$$

# R-matrices

Let $(Y, \lambda, \delta)$ be a left, respectively right, Yetter–Drinfel'd f-object over Hopf f-object $H$. Then

$$\rho = (\lambda \otimes \mathrm{id}_Y)(\mathrm{id}_H \otimes \tau_{Y,Y})(\delta \otimes \phi_Y),$$

respectively

$$\rho = (\phi_Y \otimes \lambda)(\tau_{Y,Y} \otimes \mathrm{id}_H)(\mathrm{id}_Y \otimes \delta),$$

is an R-matrix, that is a solution of the following braid group type Yang–Baxter relation in the automorphism group $\mathrm{Aut}(Y \otimes Y \otimes Y)$:

$$(\rho \otimes \mathrm{id}_Y)(\mathrm{id}_Y \otimes \rho)(\rho \otimes \mathrm{id}_Y) = (\mathrm{id}_Y \otimes \rho)(\rho \otimes \mathrm{id}_Y)(\mathrm{id}_Y \otimes \rho).$$

Moreover, this R-matrix is rigid if $Y$ is rigid.

# Hopf functorial objects as Yetter–Drinfel'd functorial objects

Let $\Delta^{(2)}$ and $\nabla^{(2)}$ be twice iterated coproduct and product.

## Proposition

*For any Hopf f-object $H\colon \mathbb{Z} \to \mathcal{C}$, the triple $(H, \nabla, \delta)$ where*

$$\delta := (\nabla \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \tau_{H,H})(\mathrm{id}_{H\otimes H} \otimes S\phi_H)\Delta^{(2)}$$

*is a left Yetter–Drinfel'd f-object over $H$.*

## Proposition

*For any Hopf f-object $H\colon \mathbb{Z} \to \mathcal{C}$, the triple $(H, \lambda, \Delta)$ where*

$$\lambda := \nabla^{(2)}(S\phi_H \otimes \mathrm{id}_{H\otimes H})(\tau_{H,H} \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes \Delta)$$

*is a right Yetter–Drinfel'd f-object over $H$.*

# Tensor algebras

Let now $\mathcal{C} \subset \mathbf{Vect}_{\mathbb{F}}$ be a braided sub-category. In this case, a Hopf functorial object and a Yetter–Drinfel'd functorial object will be respectively called a *Hopf functorial algebra* and *Yetter–Drinfel'd functorial module*.

For any $V \in \mathrm{Ob}(\mathcal{C})$, the tensor algebra $T(V)$ is a braided Hopf algebra where all elements of $V$ are primitive. We say the tensor algebra $T(V)$ is of *rank $n$* if $\dim(V) = n$. If the braiding $(\tau_{\mathcal{C}})_{V,V}$ is diagonal with respect to a linear basis $B \subset V$, then the braided Hopf algebra $T(V)$ is called of *diagonal type*. It is $\mathbb{Z}^n_{\geq 0}$-graded and admits braided Hopf algebra *scaling automorphisms*

$$\phi_t b = t_b b, \quad \forall b \in B, \quad t \colon B \to \mathbb{F}_{\neq 0}.$$

We obtain a Hopf functorial algebra $H \colon \mathbb{Z} \to \mathcal{C}$ defined by $H(*) = T(V)$ and $H(1) = \phi_t$.

# Nichols algebras

A *Nichols algebra* over a braided vector space $V \in \mathrm{Ob}\,\mathcal{C}$ is the quotient braided Hopf algebra $\mathfrak{B}(V) = T(V)/J$ over the maximal (braided) Hopf algebra ideal $J$ trivially intersecting the part $\mathbb{F} \oplus V \subset T(V)$. Nilpotent Borel parts of Lustig's small quantum groups constitute an important class of finite dimensional braided Hopf algebras in Heckenberger's classification of Nichols algebras of diagonal type. Combined with scaling automorphisms, we obtain a large class of rigid mutiparametric R-matrices. But there are many more examples in Heckenberger's list which do not come from classical Lie algebras.

- Left Yetter–Drinfel'd f-modules: look for a vector subspace $W \subset \mathfrak{B}(V)$ of elements $x$ with $\delta x = 1 \otimes x$. Then, take the quotient space $\mathfrak{B}(V)/\mathfrak{B}(V)W$.
- Right Yetter–Drinfel'd f-modules: look for *maximal vectors*: $\lambda(x \otimes y) = 0$, $y \in \mathfrak{B}(V)$. It suffices to solve $n$ equations for rank $n$ algebra:

$$\lambda(x \otimes x_i) = xx_i - t_i q(x, x_i) x_i x = 0, \quad i \in \{1, \dots, n\}.$$

# Rank 1 tensor algebra

$T(\mathbb{F}) = \mathbb{F}[x]$ with braiding $\tau(x \otimes x) = qx \otimes x$,

$$\Delta x = x_1 + x_2, \quad x_1 := x \otimes 1, \quad x_2 := 1 \otimes x, \quad x_2 x_1 = q x_1 x_2,$$

$$\Delta x^k = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q x^{k-m} \otimes x^m$$

with the $q$-binomial coefficients

$$\begin{bmatrix} k \\ m \end{bmatrix}_q := \frac{(q)_k}{(q)_{k-m}(q)_m}, \quad (q)_n := (q;q)_n = \prod_{i=1}^{n} (1 - q^i).$$

The antipode

$$Sx^k = (-1)^k q^{k(k-1)/2} x^k$$

and the scaling automorphism

$$\phi_t x^k = t^k x^k.$$

# Left Yetter–Drinfel'd f-module

Left coaction

$$\delta x^k = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^m; q)_{k-m} x^{k-m} \otimes x^m$$

$$(x; q)_k := \prod_{i=0}^{k-1} (1 - xq^i).$$

R-matrix

$$\rho(x^k \otimes x^l) = \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_q (tq^{k-m}; q)_m (tq^{k-m})^l x^{l+m} \otimes x^{k-m}.$$

When $q$ is a root of unity of order $N \in \mathbb{Z}_{>1}$. Then, the primitive element $x^N$ generates a Hopf ideal of $\mathbb{F}[x]$ with finite-dimensional quotient (Nichols) algebra $\mathbb{F}[x]/(x^N)$. The R-matrix in this case coincides with the R-matrix of Akutsu–Deguchi–Ohtzuki (ADO) polynomial.

# Right Yetter–Drinfel'd f-module

Right action

$$\lambda(x^m \otimes x^n) = (tq^m; q)_n x^{m+n}.$$

R-matrix

$$\rho(x^k \otimes x^l) = \sum_{m=0}^{l} \begin{bmatrix} l \\ m \end{bmatrix}_q (tq^k)^{l-m} (tq^k; q)_m x^{l-m} \otimes x^{k+m}.$$

# The case of generic $q$

- left Yetter–Drinfel'd f-modules: Choosing $t = q^{1-N}$ with $N \in \mathbb{Z}_{>0}$, we obtain $\delta x^N = 1 \otimes x^N$. We have left Yetter–Drinfel'd f-module $\mathbb{F}[x]/(x^N)$ of dimension $N$. The corresponding R-matrix is the one of the $N$-colored Jones polynomial.
- right Yetter–Drinfel'd f-modules: Choosing $t = q^{1-N}$ with $N \in \mathbb{Z}_{>0}$, we have a maximal element $x^{N-1}$ which generates an $N$-dimensional right Yetter–Drinfel'd f-submodule of $\mathbb{F}[x]$ with linear basis $x^k$, $0 \leq k \leq N-1$.

# An example of rank 2 Nichols algebra

Nichols algebra of diagonal type $\mathfrak{B}(V) = \tilde{H}_\omega$, $\dim V = 2$, $\tau(x_i \otimes x_j) = q_{i,j} x_j \otimes x_i$ with respect to a basis $\{x_1, x_2\} \subset V$ with the braiding matrix of the form

$$(q_{i,j}) = \begin{pmatrix} -1 & q_{1,2} \\ q_{2,1} & -1 \end{pmatrix}, \quad \omega := q_{1,2} q_{2,1}.$$

For generic values of $\omega$, the algebra is presented by the relations $x_1^2 = x_2^2 = 0$ with linear basis given by alternating words in letters $x_1$ and $x_2$.

# The case of roots of unity

Let $\omega$ be a root of unity of order $N$. Then, the primitive element $c_N := (x_2x_1)^N + (-q_{2,1}x_1x_2)^N$ generates a Hopf ideal $I := \tilde{H}_\omega c_N \tilde{H}_\omega$ with $4N$-dimensional quotient (Nichols) algebra $H_\omega = \tilde{H}_\omega / I$.
By taking the scaling automorphism $\phi_t$, $\phi_t x_i = t_i x_i$, $i \in \{1, 2\}$, we obtain a two-parameter R-matrix corresponding to the left Yetter–Drinfel'd f-module structure on $H_\omega$.
We get a knot invariant $M_K(H_\omega, t_1, t_2) \in \mathrm{End}(H_\omega)$.

### Conjecture

There exists a Laurent polynomial (2-variable ADO)

$$Q_K(\omega, t_1, t_2) \in \mathbb{Z}[\omega, t_1, t_1^{-1}, t_2, t_2^{-1}]$$

such that

$$M_K(H_\omega, t_1, t_2) = Q_K(\omega, t_1, t_2)\,\mathrm{id}_{H_\omega}\,.$$

# The case $\omega = -1$

Eight-dimensional Nichols algebra $H_{-1}$ (nilpotent Borel part of small $U_q(\mathfrak{sl}_3)$ at $q = \sqrt{-1}$).
The Laurent polynomial $Q_K(-1, t_1^2, t_2^2)$ coincides with the polynomial $\Delta_{\mathfrak{sl}_3}(t_1, t_2)$ (M. Harper, 2020, arXiv:2008.06983).

## Conjecture

There exists a polynomial $P_K(u, w) \in \mathbb{Z}[u, w]$ such that

$$Q_K(-1, t_1, t_2) = P_K(u(t_1, t_2), w(t_1, t_2)),$$

$$u(t_1, t_2) = \mathrm{zh}(t_1) + \mathrm{zh}(t_2) - \mathrm{zh}(t_1 t_2) - 2,$$

$$w(t_1, t_2) = \mathrm{zh}(t_1^2 t_2) + \mathrm{zh}(t_1 t_2^2) - \mathrm{zh}(t_1/t_2) - 2, \quad \mathrm{zh}(x) := x + x^{-1}.$$

It does not distinguish mirror images, and the special case $u = 0$ reproduces the Alexander–Conway polynomial

$$P_K(0, z^2) = \nabla_K(z).$$

# Polynomial $P_K(u, w)$ for some knots

| $K$ | $P_K(u,w)$ |
|---|---|
| $3_1$ | $1 + 4u + u^2 + w$ |
| $4_1$ | $1 - 6u + u^2 - w$ |
| $5_1$ | $1 + 12u + 19u^2 + 8u^3 + u^4 + (3 + 7u + 3u^2)w + w^2$ |
| $5_2$ | $1 + 10u + 6u^2 + 2w$ |
| $6_1$ | $1 - 10u + 6u^2 - 2w$ |
| $6_2$ | $1 - 8u - 15u^2 + 2u^3 + u^4 + (-1 - 9u + u^2)w - w^2$ |
| $6_3$ | $1 + 2u + 15u^2 + 6u^3 + u^4 + (1 + 9u + u^2)w + w^2$ |
| $7_4$ | $(1 + 2u)(1 + 18u) + 4w$ |
| $8_8$ | $1 + 10u + 36u^2 + 28u^3 + 6u^4 + 2(1 + 9u + 3u^2)w + 2w^2$ |
| $9_2$ | $1 + 20u + 24u^2 + 4w$ |
| $10_{129}$ | $1 + 10u + 32u^2 + 36u^3 + 6u^4 + 2(1 + 8u + 2u^2)w + 2w^2$ |
| $11n34$ | $1 + 12u + 8u^2 + 60u^3 + 48u^4 + 8u^5$<br>$+2u(1 + 2u)(-1 + 6u)w + 2u^2w^2$ |
| $11n42$ | $1 + 12u + 8u^2 - 12u^3 - 2uw$ |
| $11n73$<br>$11n74$ | $1+20u+10u^2+4u^3+u^4+2(1+4u+u^2)w+w^2$ |

# The case of generic $\omega$

## Finite-dimensional right Yetter–Drinfel'd f-modules over $\tilde{H}_\omega$

*Bi-degree* $= \mathbb{Z}^2_{\geq 0}$-degree of $\tilde{H}_\omega$.
*Degree* $= \mathbb{Z}_{\geq 0}$-degree $=$ sum of the components of bi-degree.
There are no maximal vectors of odd degree.
Maximal vectors of even degree:

$$v_{n,\alpha} = (x_1 x_2)^n + \alpha (x_2 x_1)^n, \quad \alpha \in \mathbb{F}$$

### Proposition

*Assume $1 \notin \omega^{\mathbb{Z}_{>1}}$. Then $v_{n,\alpha}$ is a maximal vector if and only if*

$$\alpha = t_2(-q_{1,2})^n, \quad t_1 t_2 \omega^n = 1$$

*and, if $(1-t_1)(1-t_2) \neq 0$, the right Yetter–Drinfel'd f-module $Y_n$ generated by $v_{n,\alpha}$ is $4n$-dimensional given by the linear span of $v_{n,\alpha}$ and all vectors of degree $\leq 2n-1$.*

# Two-variable coloured Jones polynomials

For any $n \in \mathbb{Z}_{\geq 1}$ we obtain an R-matrix over the $4n$-dimensional vector space $Y_n$ which produces a knot invariant $W_K(Y_n, \omega, t_1) \in \mathrm{End}(Y_n)$.

**Conjecture**

There exists a Laurent polynomial

$$Q_{n,K}(\omega, t_1) \in \mathbb{Z}[\omega, \omega^{-1}, t_1, t_1^{-1}] \tag{1}$$

such that

$$W_K(Y_n, \omega, t_1) = Q_{n,K}(\omega, t_1)\, \mathrm{id}_{Y_n} . \tag{2}$$

Calculations for $n = 1$ indicate that $Q_{1,K}(\omega, t_1)$ coincides with the Links–Gould two-variable knot polynomial coming from the quantum super algebra $U_q(gl(2|1))$.