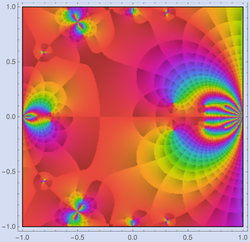


The 3D index and Dehn surgery

joint work with C. Hodgson and H. Rubinstein

Moduli & Friends seminar

Daniele Celoria

Overview

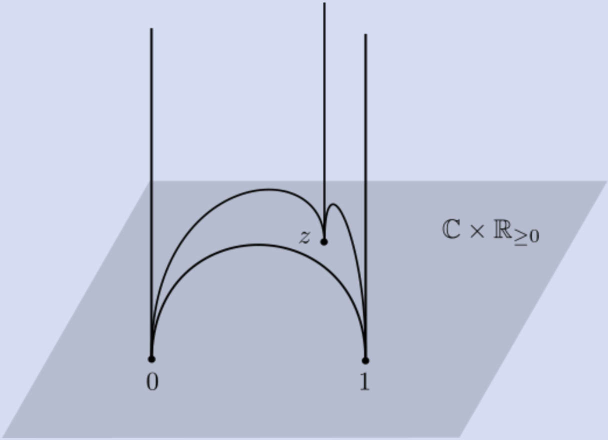
Idea: rigorous proof of a Dehn surgery formula for the 3D index.

- hyperbolic geometry in dimension 3
- the DGG 3D index
- Dehn surgery
- the proof
- applications & open questions

Introduction

In what follows, $M =$ compact, connected, orientable 3-manifold with $\partial M = c$ tori (*e.g.* link complement).

M can be described via an ideal triangulation:



Thurston's Gluing equations

Edge equations (how tetrahedra fit together)

$$\sum_{i=1}^{n} \log(z_i) = 2\pi i$$

Completeness equations (holonomy)

$$\forall \sigma \subset \partial M \ \ \sum_{i} \epsilon_i \log(z_i) = 0$$

Can be read off SnapPy's `gluing_equations()` command.

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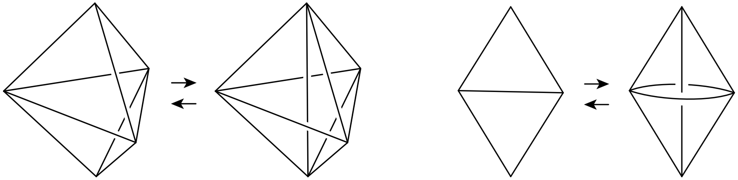
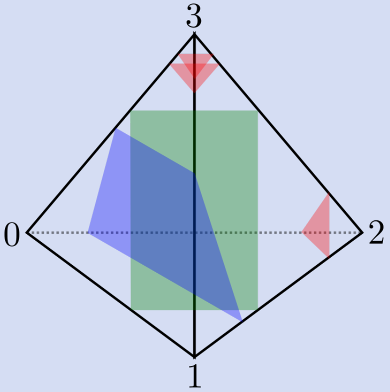
Pachner moves



Normal surfaces

Surfaces intersecting each tetrahedron in a ‘straight’ disk.



Can be described as $\mathbb{N}$-solutions of a linear system in $(\mathbb{R}^{3+4})^n$.

3D index

The **3D-index** is an 'invariant' of cusped 3-manifolds introduced by the physicists Dimofte-Gaiotto-Gukov in 2011, which contains much information about the topology/geometry of the manifold.

$$M \rightsquigarrow \mathcal{T}_M + \text{'boundary data' } \gamma \rightsquigarrow I^{\gamma}_{\mathcal{T}_M}(q) \in \mathbb{Z}[\![q^{\frac{1}{2}}]\!]$$

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Encodes and detects many geometric/topological properties of M:

- hyperbolicity
- is a “generating function” over normal surfaces
- coefficients related to the topology of M
- connections with $\hat{A}$ polynomials

Definition

The *tetrahedral index* $I_\Delta \colon \mathbb{Z}^2 \to \mathbb{Z}[\![q^{\frac{1}{2}}]\!]$ is defined by

$$I_\Delta(m,e)(q) = \sum_{n=\max\{0,-e\}}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)-(n+\frac{1}{2}e)m}}{(q;q)_n (q;q)_{n+e}} \quad \text{for}$$

where $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$ is a q-*Pochhammer symbol*.

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A more symmetric version is given by

$$J_\Delta(a,b,c) = (-\sqrt{q})^{-b} I_\Delta(b-c, a-b) \text{ for } a,b,c \in \mathbb{Z},$$

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Remark

$$I_\Delta(m,e) = \frac{q^{-\frac{me}{2}}(q^{e+1};q)_\infty}{(q;q)_\infty} {}_1\phi_1 \left[\begin{matrix} 0 \\ q^{e+1} \end{matrix} ; q, q^{1-m} \right].$$

(In this form it is a specialised Hahn-Exton q-Bessel function)

q-hypergeometric series

Definition

For $0 < q < 1$, define

$$ {}_r\phi_s \left[\begin{matrix} a_1 & \dots & a_r \\ b_1 & \dots & b_s \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s, q; q)_n} \left((-1)^n q^{\binom{n}{2}} \right)^{1+s-r} z^n $$

where

$$ (a;q)_n = \prod_{i=0}^{n-1} (1 - aq^i) \text{ and } (a_1, \dots, a_m; q)_n = \prod_{i=1}^{m} (a_i; q)_n. $$

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Example:

$$ e_q(z) = \sum_{n \geq 0} \frac{z^n}{(q;q)_n} = {}_1\phi_0 \left[\begin{matrix} 0 \\ - \end{matrix}; q, z\right] = \frac{1}{(z;q)_\infty} \text{ then } \lim_{q \to 1^-} e_q(z) = e^z $$

Some useful identities

- *symmetry* $I_\Delta(m,e) = I_\Delta(-e,-m)$
- *3 term relations*

$$\begin{aligned} q^{\frac{e}{2}} I_\Delta(m+1,e) + q^{-\frac{m}{2}} I_\Delta(m,e+1) - I_\Delta(m,e) = 0 \\ q^{\frac{e}{2}} I_\Delta(m-1,e) + q^{-\frac{m}{2}} I_\Delta(m,e-1) - I_\Delta(m,e) = 0 \end{aligned} \quad (1)$$

- *quadratic identity*

$$\sum_{e\in\mathbb{Z}} I_\Delta(m,e) I_\Delta(m,e+c) q^e = \delta_{c,0},$$

- *pentagon identity*

$$\sum_{e\in\mathbb{Z}} q^e I_\Delta(m_1, e+x_1) I_\Delta(m_2, e+x_2) I_\Delta(m_1+m_2, e+x_3) =$$
$$q^{-x_3} I_\Delta(m_1 - x_2 + x_3, x_1 - x_3) I_\Delta(m_2 - x_1 + x_3, x_2 - x_3).$$

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Definition

$$I^0_{\mathcal{T}}(q) = \sum_{\substack{k:\mathcal{E}\to\mathbb{Z}, \\ k|_{\mathcal{E}'}=0}} q^{k(\mathcal{E})} \prod_{j=1}^{n} J_\Delta(q_j(k))$$

where $k(\mathcal{E}) = \sum_{e\in\mathcal{E}} k(e)$ and $\mathcal{E}'$ is a suitable choice of r edges. (For $r = 1$ any edge is suitable.)

More generally:

$$I_{\mathcal{T}}^{\gamma}(q) = \sum_{\substack{k:\mathcal{E}\to\mathbb{Z},\\ k|_{\mathcal{E}'}=0}} q^{k(\mathcal{E})} \prod_{j=1}^{n} J_{\Delta}(q_j(k) + \gamma_j)$$

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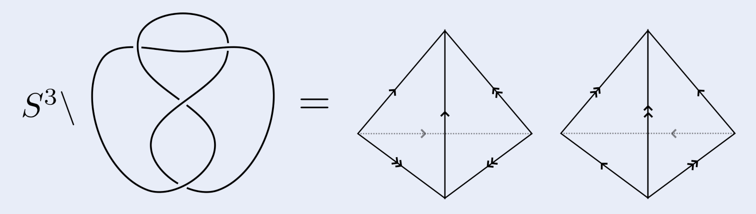
- Physics predicts that $I_{\mathcal{T}}$ should be a topological invariant of M. However, it is not defined for all triangulations $\mathcal{T}$!

Example: complement of 4_1



edge/curve	weight	a_1	b_1	c_1	a_2	b_2	c_2
e_1	k	2	1	0	2	1	0
e_2	0	0	1	2	0	1	2
2μ	x	0	0	1	-1	0	0
2λ	$\frac{1}{2}y$	0	0	0	2	0	-2

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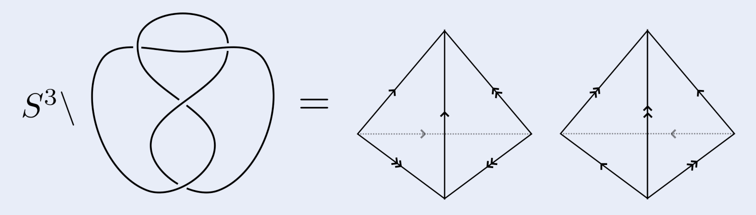

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Hence, for $x, y \in \mathbb{Z}$,

$$I_{\mathcal{T}}^{(x,y)}(q) = \sum_{k\in\mathbb{Z}} q^k J_\Delta(2k,k,x) J_\Delta(2k-x+y,k,-y)$$

$$= \sum_{k\in\mathbb{Z}} I_\Delta(k-x,k) I_\Delta(k+y,k-x+y).$$

Some results

[S. Garoufalidis, C. Hodgson, N. Hoffmann, J.H. Rubinstein, H. Segerman]

- $I^{\gamma}_{\mathcal{T}}(q)$ is well-defined (for all γ) iff $\mathcal{T}$ is *1-efficient* (i.e. no embedded normal spheres and tori except peripheral tori.)

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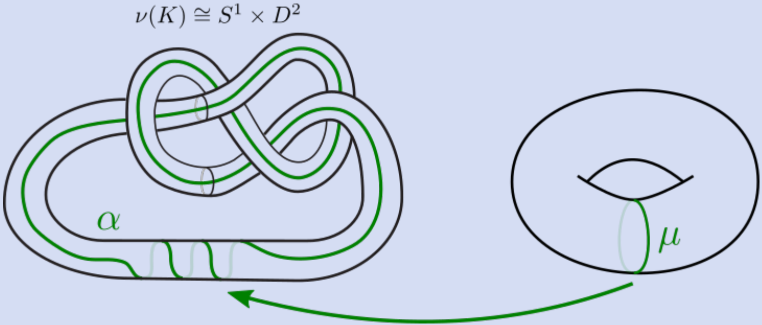
- $I^\gamma_\mathcal{T}(q)$ is invariant under 2-3 moves, 0-2 moves *if all triangulations are 1-efficient*.
- If M is hyperbolic we get a *topological invariant* $I^\gamma_M(q)$.
- Can write $I^\gamma_\mathcal{T}(q)$ as a sum over *singular Q-normal surfaces* in $\mathcal{T}$ with boundary γ.

Dehn surgery



3D-index and Dehn filling

In 2018, the physicists D. Gang and K. Yonekura proposed a formula giving the 3D-index for Dehn fillings on a cusped 3-manifold M.

The Gang-Yonekura formula gives the 3D-index for $M(\alpha)$ *as an infinite linear combination of the 3D-indices* $I_M^{\gamma}(q)$ *for* M *over boundary classes* γ *having intersection number* 0 *or* ± 2 *with* α*.*

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If $K = \ker(H_1(T;\mathbb{Z}) \to H_1(M;\mathbb{Z}_2))$, $|\gamma| = \#$ components of γ:

$$I_{M(\alpha)}(q) = \frac{1}{2}\left(\sum_{\substack{\gamma \in K \\ \gamma\cdot\alpha=0}} (-1)^{|\gamma|}(q^{\frac{|\gamma|}{2}} + q^{-\frac{|\gamma|}{2}}) I_M^\gamma(q) - \sum_{\substack{\gamma \in K \\ \gamma\cdot\alpha=\pm 2}} I_M^\gamma(q) \right)$$

Main result

We wanted to show that this formula holds for Dehn fillings on some (but not all!) cusps of a multi-cusped manifold.

Theorem [CHR]

Let M be a compact orientable 3-manifold with boundary consisting of at least 2 tori, and let T be one component of ∂M. Let $\mathcal{T}_M$ be a *1-efficient triangulation* with a *standard cusp* at T.

Given a surgery curve $\alpha \subset T$, let $\mathcal{T}(\alpha)$ be the ideal triangulation of $M(\alpha)$ obtained from $\mathcal{T}$ by replacing the standard cusp by the *layered solid torus* with α bounding a meridian disc.
Then the Gang-Yonekura formula holds for the triangulations $\mathcal{T}$ and $\mathcal{T}(\alpha)$ for all α with at most 3 exceptions.

Corollary [CHR]

Let M be a cusped hyperbolic 3-manifold with at least two cusps, and assume that one cusp T has a *generic shape*, i.e. the corresponding Euclidean torus is not rectangular. Then the Gang-Yonekura formula holds for almost all Dehn fillings $M(\alpha)$.

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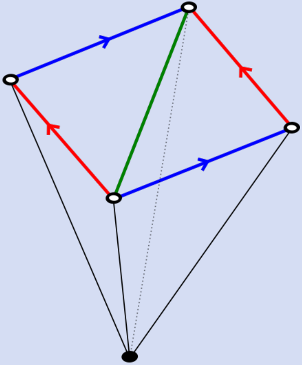
Proof: By first expanding the cusp at T until it bumps into itself, then expanding the other cusps, we obtain a canonical (Epstein-Penner) ideal triangulation $\mathcal{T}$ with a standard cusp at T. Then work of Guéritaud-Schleimer shows that for almost all α, the corresponding canonical triangulation of $M(\alpha)$ is isomorphic to the triangulation $\mathcal{T}(\alpha)$.
So we obtain $I_M = I_{\mathcal{T}}$ and $I_{M(\alpha)} = I_{\mathcal{T}(\alpha)}$. $\square$

Explanation of the theorem

A *standard cusp* has a triangulation by two ideal tetrahedra:

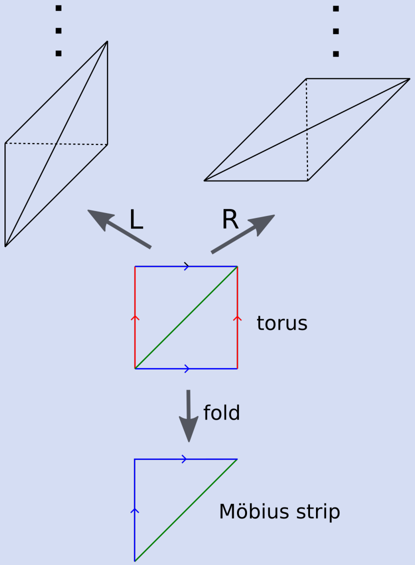


Theorem [Howie-Mathews-Purcell] (paraphrased)

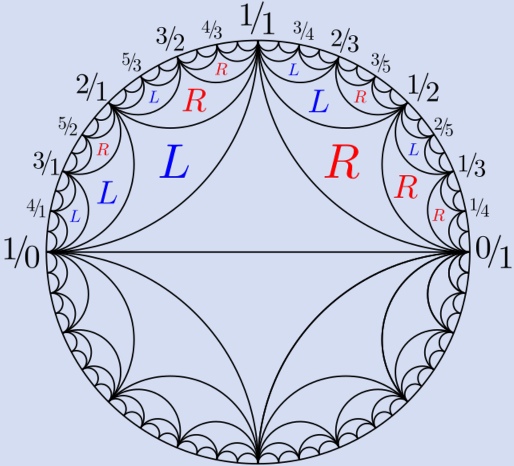
If there's more than one cusp, there exists a triangulation where all but one cusps are standard.

Layered solid tori

- start with two triangles giving a torus,
- add layers of tetrahedra on top to get a triangulation of $T^2 \times [0,1]$,
- "fold up" the bottom torus $T^2 \times 0$ onto a Möbius strip.
- This gives a solid torus (which is a neighbourhood of the Möbius strip).

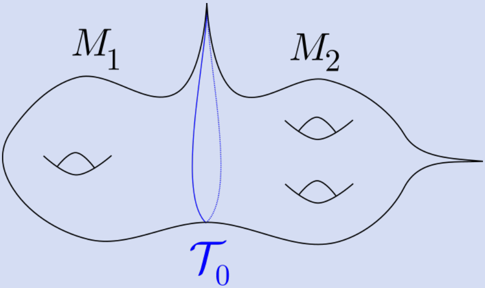


Layered triangulations of the solid torus are parameterised by finite words in the alphabet $\{L, R\}$ giving triangles in the Farey tessellation (related to continued fraction expansion).



Gluing formulas and relative index

Let $\mathcal{T}$ be an ideal triangulation of M and $\mathcal{T}_0$ be a union of triangles in $\mathcal{T}$ splitting $\mathcal{T}$ into triangulations $\mathcal{T}_1, \mathcal{T}_2$ of submanifolds M_1, M_2 respectively.



Let $\mathcal{E}$ be the set of all edges of $\mathcal{T}$,
$\mathcal{E}'$ a collection of r edges "omitted" in computing the index of $\mathcal{T}$,
$\mathcal{E}_i$ the set of edges in $\mathcal{T}_i$, and $\mathcal{E}'_i = \mathcal{E}_i \cap \mathcal{E}'$.

Relative index

For any prescribed $\gamma \in K \subset H_1(\partial M;\mathbb{Z})$ and "boundary weights" $b_0 : \mathcal{E}_0 \to \mathbb{Z}$ with $b_0|_{\mathcal{E}_0'} = 0$ we define *relative indices* for $i = 1, 2$ by

$$I^{\gamma}_{\mathcal{T}_i}(q; b_0) = \sum_{\substack{k:\mathcal{E}_i \to \mathbb{Z},\\ k|_{\mathcal{E}_i'}=0,\\ k|_{\mathcal{E}_0}=b_0}} q^{k(\mathcal{E}_i \setminus \mathcal{E}_0)} \prod_{j=1}^{n} J_\Delta(q_j(k) + \gamma_j)$$

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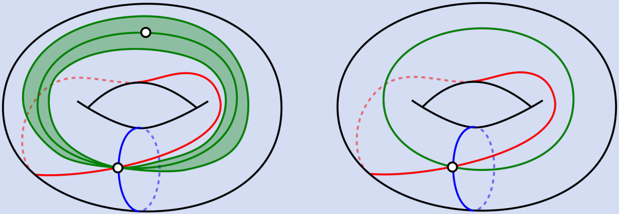
Gluing formula

$$I^{\gamma}_{\mathcal{T}}(q) = \sum_{\substack{b_0:\mathcal{E}_0 \to \mathbb{Z}, \\ b_0|\mathcal{E}_0'=0}} q^{b_0(\mathcal{E}_0)} \, I^{\gamma}_{\mathcal{T}_1}(q; b_0) \, I^{\gamma}_{\mathcal{T}_2}(q; b_0)$$

where $b_0(\mathcal{E}_0) = \sum_{e \in \mathcal{E}_0} b_0(e)$.

Local version



By the gluing formula, it suffices to show:
relative index of a layered solid torus = Gang-Yonekura formula applied to the relative index of a standard cusp.

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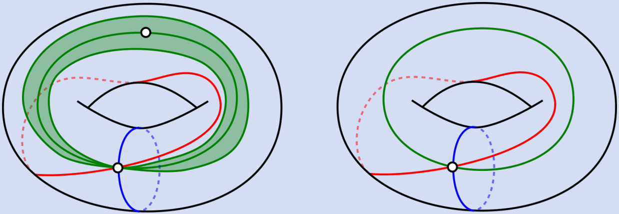

By the gluing formula, it suffices to show:
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Proof of this local version:

- Use induction on the number of tetrahedra in the solid torus.
- Inductive step is proved using the pentagon identity.
- The base case (much harder for us!) needs some new algebraic identities.

Base case: $\mathsf{LST}(1,1,2)$

Theorem [CHR]

For every $b=(b_1,b_2)\in\mathbb{Z}^2$,

$$\frac{1}{2}\sum_{k\in\mathbb{Z}}(q^k+q^{-k})G_b(k,k)-G_b(k,k+1)-G_b(k,k-1)=q^{-b_1}\delta_{b_1,b_2}$$

where $G_b(e,m)=I_\Delta(e-b_1,m+b_2)I_\Delta(-e-b_1,-m+b_2)$.

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Theorem [CHR]

For every $b=(b_1,b_2)\in\mathbb{Z}^2$,

$$\frac{1}{2}\sum_{k\in\mathbb{Z}}(q^k+q^{-k})G_b(k,k)-G_b(k,k+1)-G_b(k,k-1)=q^{-b_1}\delta_{b_1,b_2}$$

where $G_b(e,m)=I_\Delta(e-b_1,m+b_2)I_\Delta(-e-b_1,-m+b_2)$.

Sketch of Proof: First introduce the "diagonal" generating functions:

$$\varphi_r(z,q)=\sum_{e\in\mathbb{Z}}I_\Delta(e-r,e)z^e,\ \text{where } r\in\mathbb{Z}.$$

Then the LHS in the theorem is the coefficient of z^0 in

$$z^{-2b_2}\left(\varphi_r(zq^{\frac{1}{2}},q)\varphi_r(zq^{-\frac{1}{2}},q)-\varphi_{r+1}(z,q)\varphi_{r-1}(z,q)\right),$$

where $r=b_1+b_2$.

A meromorphic approach

To study $\varphi_r(z, q)$, we begin with Garoufalidis-Kashaev's meromorphic 3D index:

Definition

$$\psi^0(z, w, q) = c(q) \frac{(z^{-1}; q)_\infty}{(-qz; q)_\infty} \frac{(-qw; q)_\infty}{(w^{-1}; q)_\infty} \frac{(-qzw^{-1}; q)_\infty}{(wz^{-1}; q)_\infty},$$

where $c(q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}$.

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Theorem [Garoufalidis-Kashaev]

$$\psi^0(z,w,q) = \sum_{e,m\in\mathbb{Z}} (-q)^e I_\Delta(m,e)(q^2) z^e w^m$$

We then extract the "diagonal series" $\varphi_r(z,q)$ as a complex line integral, and use Cauchy's residue theorem to express this as an infinite sum of residues:

$$\varphi_r(z,q) = \sum_{e\in\mathbb{Z}} (-q)^e I_\Delta(e-r,e)(q^2)x^{e-r} = \frac{1}{2\pi i}\int_{|s|=\rho} \psi^0\left(s, \frac{x}{s}\right)\frac{\mathrm{d}s}{s^{r+1}}$$

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Theorem [CHR]

$\varphi_r(z,q) =$

$$\begin{cases} (-zq^{-\frac{1}{2}})^{\frac{r}{2}}\,{}_3\phi_3\left[\begin{matrix} -z^{-1}\sqrt{q} & -z\sqrt{q} & 0 \\ -q & \sqrt{q} & -\sqrt{q}\end{matrix}; q, q^{1-\frac{r}{2}}\right] & r \text{ even} \\ (-z)^{\frac{r-1}{2}} q^{\frac{1-r}{2}} \frac{(1+z)}{1-q}\,{}_3\phi_3\left[\begin{matrix} -z^{-1}q & -zq & 0 \\ -q & q^{\frac{3}{2}} & -q^{\frac{3}{2}}\end{matrix}; q, q^{\frac{3-r}{2}}\right] & r \text{ odd.}\end{cases}$$

After further manipulations we can apply a *q-analogue of the Pythagoras theorem* proved by F. Štampach to obtain:

Proposition [CHR]

For $r \in \mathbb{Z}$ and $z, q \in \mathbb{C}, |q| < 1$

$$\varphi_r(zq^{\frac{1}{2}}, q)\varphi_r(zq^{-\frac{1}{2}}, q) - \varphi_{r+1}(z, q)\varphi_{r-1}(z, q) \equiv z^r q^{-\frac{1}{2}r}$$

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Corollary

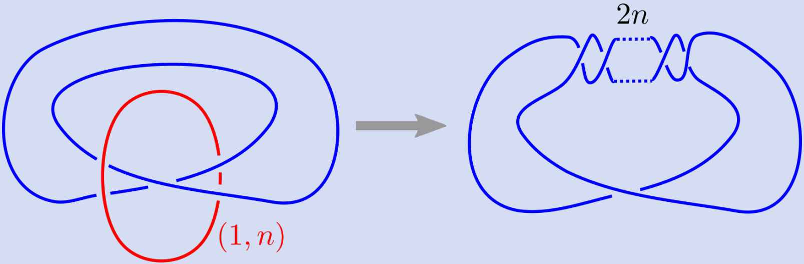
If $r \in \mathbb{Z}$ is even, then:

$$\sum_{e \in \mathbb{Z}} (-\sqrt{q})^e I_\Delta(e - r, e) = 1$$

If $r \in \mathbb{Z}$ is odd, then:

$$\sum_{e \in \mathbb{Z}} (-q)^e I_\Delta(e - r, e) = 1$$

Example: 3D-index for alternating torus knots



Using some of our identities for the tetrahedral index and applying the Gang-Yonekura formula we can show

$$I^{(x,y)}_{T(2n-1,2)}(q) = \begin{cases} 1 & \text{if } (x,y) \text{ is a multiple of } (-2(2n-1),1) \\ 0 & \text{otherwise} \end{cases}$$

Asymptotic stability formula

Let α, β be homology classes of dual simple closed curves on $T \subset \partial M$ with $\alpha \notin K, \beta \in K$, (*e.g.* $\alpha = \mu, \beta = \lambda$).

Theorem [CHR]

Let $\alpha_n = \alpha + n\beta$ for $n \in \mathbb{Z}$. Then:

$$\lim_{n \to \infty} I_{M(\alpha_n)}(q) \to I_M^0(q) - I_M^{2\beta}(q).$$

Crucially, the result depends on the 'direction' we choose, unlike in Thurston's hyperbolic filling theorem!

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Surgeries on the Whitehead link

$\frac{1}{n}$ $1 - 4q + q^2 + 16q^3 + 22q^4 + q^5 - 72q^6 - 158q^7 + \dots$

n $1 - q - q^2 + 4q^3 + 6q^4 + 5q^5 - 11q^6 - 32q^7 + \dots$

Closed manifolds?

The index is not defined for closed manifolds. Using Gang-Yonekura's formula, we can try to force a definition!

It appears to detect non-hyperbolic surgeries:

$$\begin{array}{lcl}
S_0^3(4_1) & \rightsquigarrow & 1 \\
S_1^3(4_1) & \rightsquigarrow & 1 \\
S_2^3(4_1) & \rightsquigarrow & 1 \\
S_3^3(4_1) & \rightsquigarrow & 1 \\
S_4^3(4_1) & \rightsquigarrow & \infty - 2q^2 - 2q^3 - 4q^4 - 4q^5 + \dots \\
S_5^3(4_1) & \rightsquigarrow & 1 - q - 2q^2 - q^3 - q^4 + q^5 + \dots \\
S_6^3(4_1) & \rightsquigarrow & 1 - \sqrt{q} - q^{3/2} - q^2 - q^{5/2} + q^{9/2} + \dots
\end{array}$$

(Note the infinite constant term for $(4,1)$-surgery!)

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