Size in Contact Geometry

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The Erlangen program

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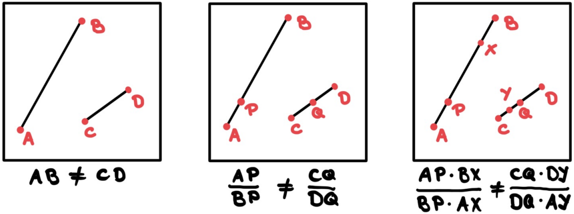

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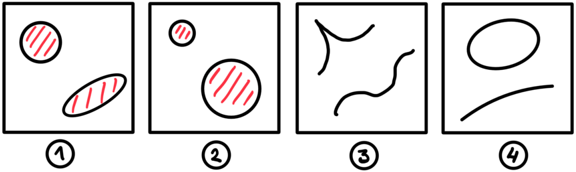
Rigidity $\longrightarrow$ Flexibility



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More specific question

Does contact geometry remember the size?

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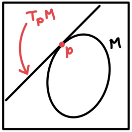

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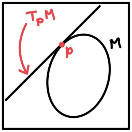

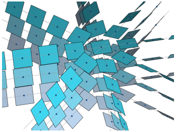

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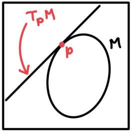

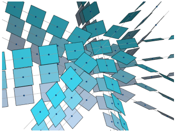

- Figure on the right: ξ° in $\mathbb{R}^3$ [Pmassot, CC BY-SA 3.0, via Wikimedia Commons]

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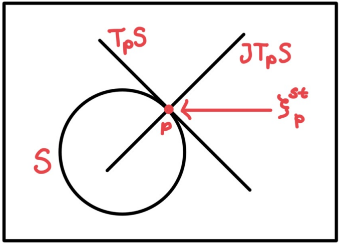

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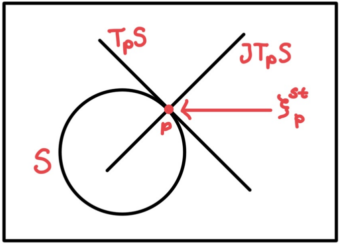

Fact

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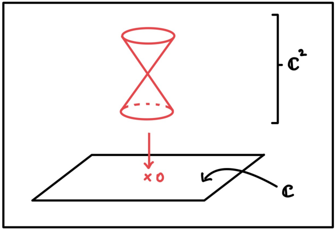

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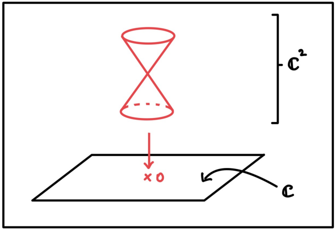

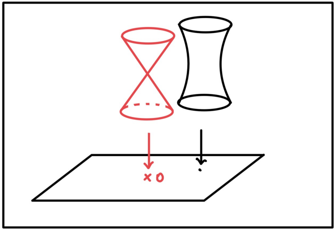

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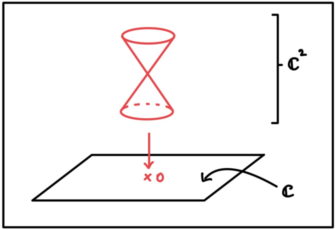

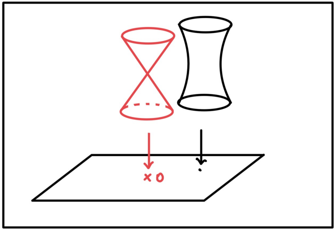

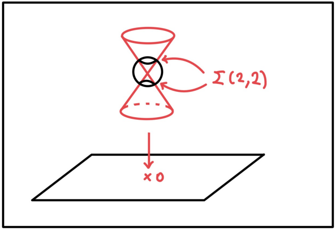

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Every contact isotopy is furnished by some contact Hamiltonian.

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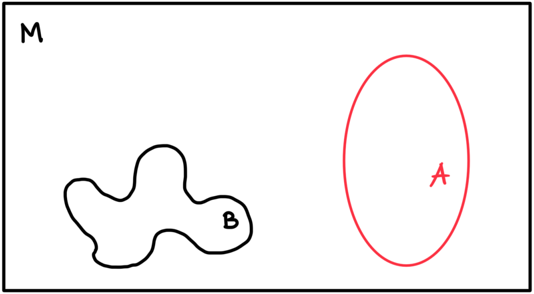

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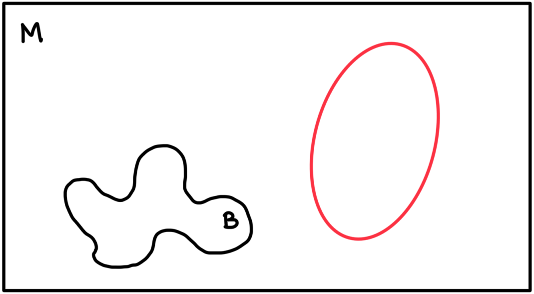

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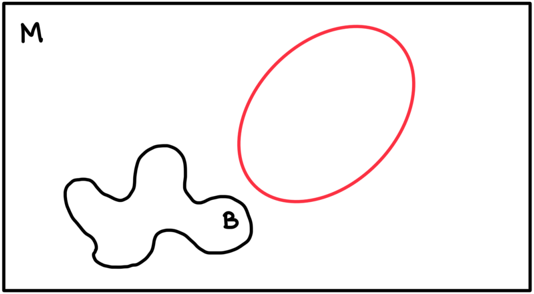

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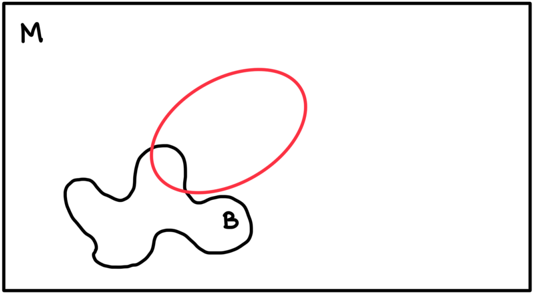

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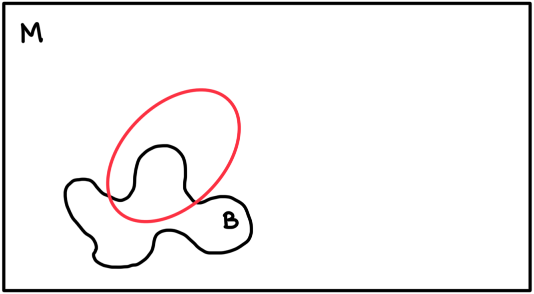

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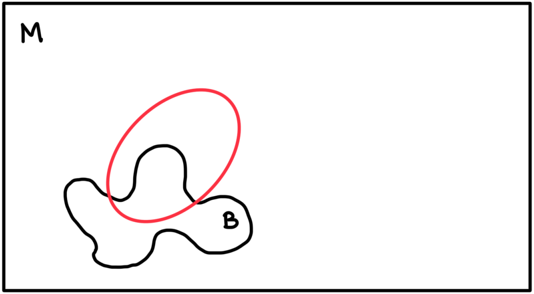

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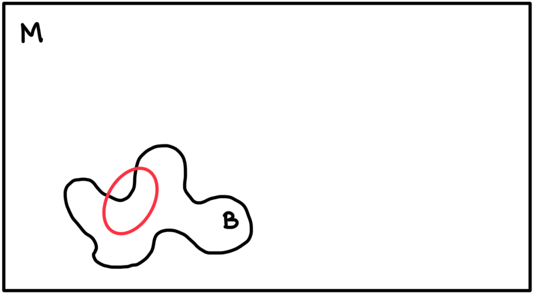

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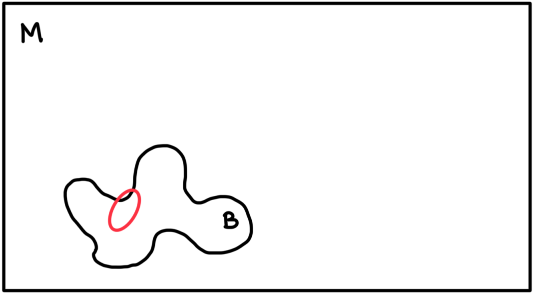

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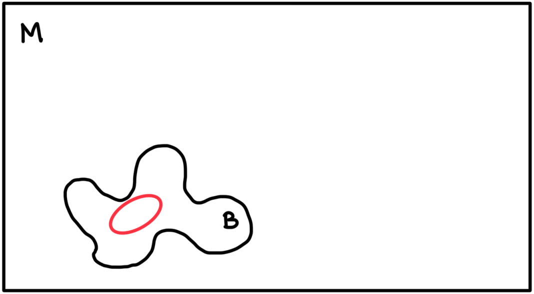

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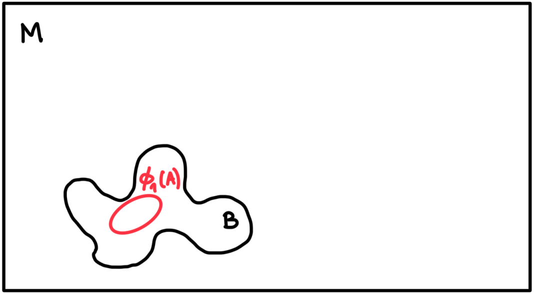

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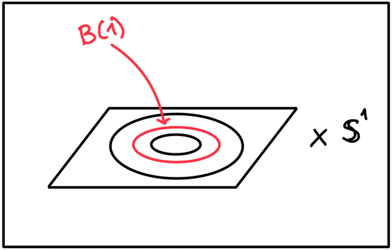

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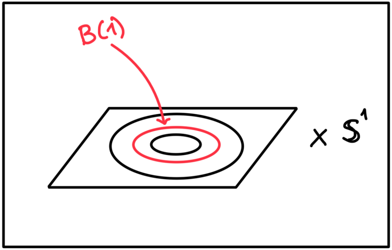

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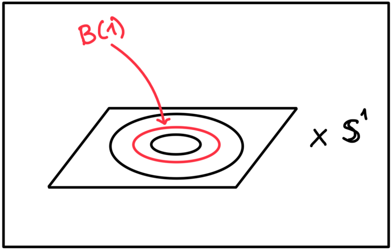

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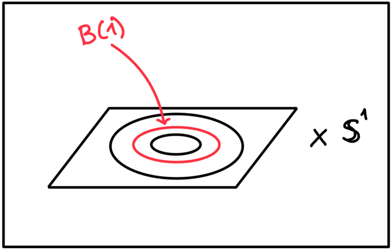

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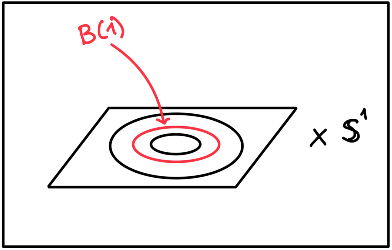

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Corollary

Contact non-squeezing on homotopy spheres is genuinely contact phenomenon.

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Theorem (Fauteux-Chapleau and Helfer)

There exist infinitely many pairwise non-contactomorphic tight contact structures on $\mathbb{R}^{2n+1}$ if $n > 1$.

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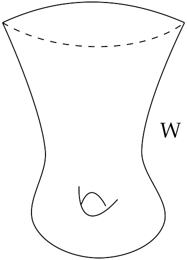
- W a Liouville domain



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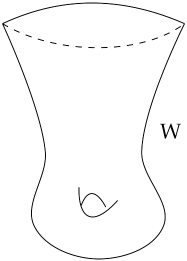

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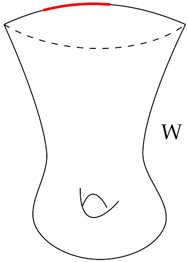

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In the situation of the proposition above, if $\Omega_a \subset \Omega_b$ are open subsets of ∂W, then the following diagram commutes:

$$\begin{array}{ccc} SH_*^{\Omega_a}(W) & \xrightarrow[\mathcal{C}(\phi)]{} & SH_*^{\phi(\Omega_a)}(W) \\ \Phi_a^b \downarrow & & \downarrow \Phi_a^b \\ SH_*^{\Omega_b}(W) & \xrightarrow{\mathcal{C}(\phi)} & SH_*^{\phi(\Omega_b)}(W). \end{array}$$

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Proposition

Let W be a Liouville domain with $\dim SH_*(W) = \infty$ and $\dim W \geqslant 4$. Then, for every $C \in \mathbb{R}$, there exists a contact Darboux chart D such that the continuation map $SH_*^{\partial W \setminus D}(W) \to SH_*(W)$ has rank greater than C.

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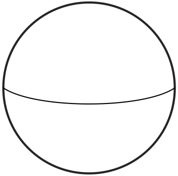

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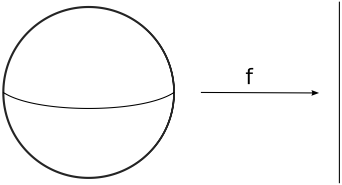

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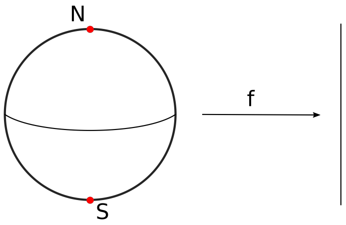

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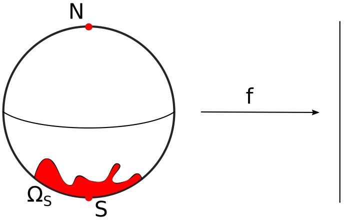

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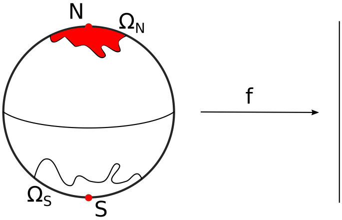

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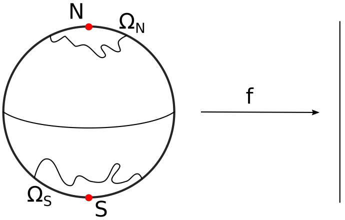

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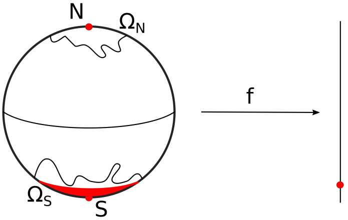

- $f^{-1}(-\infty, c] \subset \Omega_S$ for c close to $\min f$.

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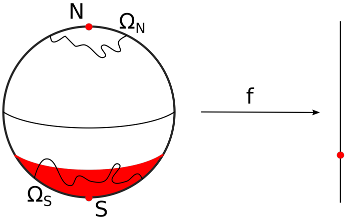

- $f^{-1}(-\infty, c] \subset \Omega_S$ for c close to $\min f$.

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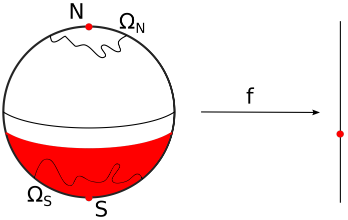

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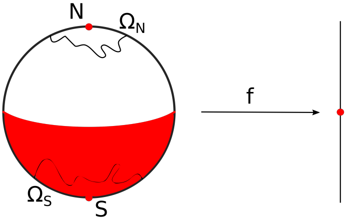

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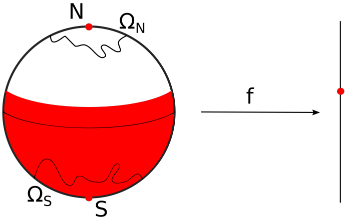

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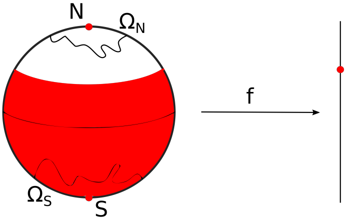

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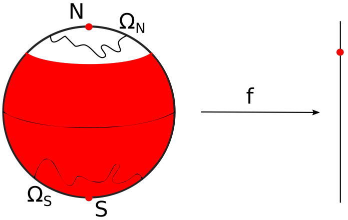

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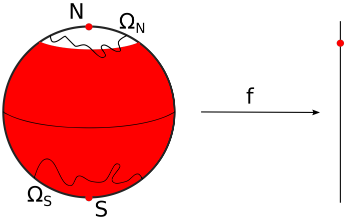

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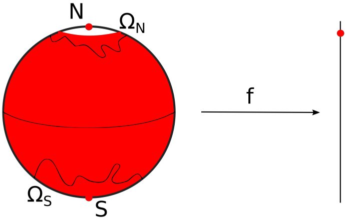

- $f^{-1}(-\infty, c] \subset \Omega_S$ for c close to $\min f$.
- $f^{-1}(-\infty, c] \supset \partial W \setminus \Omega_N$ for c close to $\max f$.

Thank you!