Size in Contact Geometry

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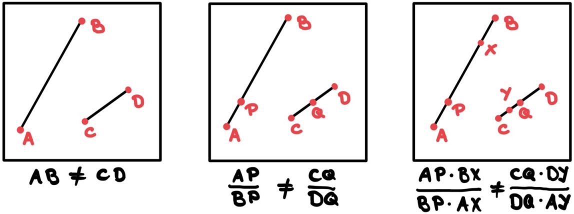
The Erlangen program

- Felix Klein : Vergleichende Betrachtungen uber neuere geometrische Forschungen, 1872
- Erlangen program : study geometry by investigating the group of transformations and their invariants

	Euclidean	Affine	Projective
length	✓	✗	✗
ratio	✓	✓	✗
cross-ratio	✓	✓	✓



Another example

A manifold with additional structures

Let M be a manifold with

- topology τ
- smooth structure (i.e. smooth atlas) $\mathcal{A}$
- measure μ
- Riemannian metric g.

- $\mathrm{Aut}(M, \tau) = \mathrm{Homeo}(M) =$ the group of homeomorphisms of M
- $\mathrm{Aut}(M, \mathcal{A}) = \mathrm{Diff}(M) =$ the group of diffeomorphisms of M
- $\mathrm{Aut}(M, \mu) =$ the group of volume preserving diffeomorphisms of M
- $\mathrm{Aut}(M, g) = \mathrm{Iso}(M)$ the group of isometries of M

Assumption

We assume the structures $\tau, \mathcal{A}, \mu,$ and g are mutually compatible.

That is . . .

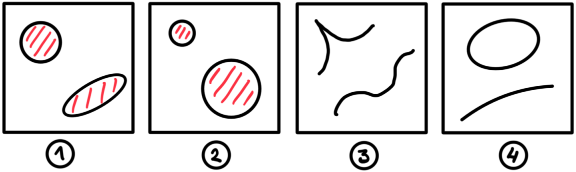
Another example

Assumption

- The topology induced by $\mathcal{A}$ is τ
- g is smooth with respect to $\mathcal{A}$
- μ is the measure induced by g.

$$\mathrm{Iso}(M) \subset \mathrm{Aut}(M, \mu) \subset \mathrm{Diff}(M) \subset \mathrm{Homeo}(M).$$

Rigidity $\longrightarrow$ Flexibility



The main character in this story

- Contact manifold (M, ξ)
- M a smooth manifold
- ξ a *contact structure* on M
- $\mathrm{Aut}(M, \xi) =: \mathrm{Cont}(M, \xi)$
- The elements of $\mathrm{Cont}(M, \xi)$ are called *contact transformations*, or shortly, *contactomorphisms*.
- $\mathrm{Diff}(M) \supset \mathrm{Cont}(M, \xi)$

Question

How far does contact geometry go beyond smooth topology?

More specific question

Does contact geometry remember the size?

What is a contact structure?

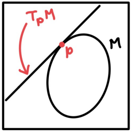
Definition

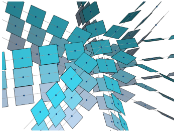
A contact structure ξ on a smooth manifold M is a smooth family of hyperplanes $\xi_p \subset T_pM, p \in M$ that locally looks like

$$\xi^\circ := \ker\left(dz + \sum_{k=1}^{n}(x_j dy_j - y_j dx_j)\right),$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $z \in \mathbb{R}$.

- T_pM is the tangent space of M at the point p.





- Figure on the right: ξ° in $\mathbb{R}^3$ [Pmassot, CC BY-SA 3.0, via Wikimedia Commons]

What is a contact structure?

Alternative definition

A contact structure on a smooth $(2n+1)$-dimensional manifold M is $\ker \alpha$ where α is a locally defined 1-form on M such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing.

- The 1-form α from the alternative definition is called *cotact form*.
- If the contact form is globally defined (i.e. everywhere on M), than we say that ξ is *cooriented* contact structure.
- In this talk, all contact structures will be cooriented.

Definition

A contactomorphism $\phi : M \to N$ between contact manifolds (M, ξ) and (N, ζ) is a diffeomorphism such that $d\phi\xi = \zeta$.

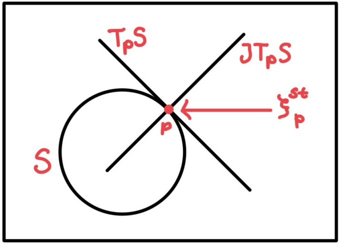
- $\mathrm{Cont}(M, \xi)$ is the group of all contactomorphisms $M \to M$.

Why do we care about contact manifolds?

- Geometrical optics
- Classical mechanics
- Thermodynamics
- Geometric quantization
- Kronheimer-Mrowka: proof of property P conjecture
- Low-dimensional topology: invariants of knots and smooth three-manifolds

Examples

- $(\mathbb{R}^{2n+1}, \xi^\circ)$
- $(\mathbb{S}^{2n+1}, \xi^{st})$ where ξ^{st} is defined as follows:
  - See $\mathbb{S}^{2n+1}$ as a subset of $\mathbb{C}^{2n+2}$
  - Denote by J the complex structure on $\mathbb{C}^{2n+2}$
  - $\xi^{st}_p := T_p\mathbb{S}^{2n+1} \cap JT_p\mathbb{S}^{2n+1}$.



Fact

The contact manifolds $(\mathbb{R}^{2n+1}, \xi^\circ)$ and $(\mathbb{S}^{2n+1} \setminus \{pt\}, \xi^{st})$ are contactomorphic.

Examples

- $\left(\mathbb{R}^{2n} \times \mathbb{S}^1, \xi\right)$ where $\xi := \ker\left(d\theta + \sum_{k=1}^{n}(x_j dy_j - y_j dx_j)\right)$
- Let M be a smooth manifold. Then, the space of 1-jets $J^1(M, \mathbb{R})$ has a natural contact structure.
  $J^1(M, \mathbb{R}) := C^\infty(M)/\sim$, where $f \sim g$ iff f and g have the same derivatives up to order 1.

Theorem (Martinet, 1971)

Every compact, orientable 3-dimensional manifold admits a contact structure.

- Let (M, g) be a Riemannian manifold. The, the unit cotangent bundle S^*M has a natural contact structure.
  - $T^*M := \bigcup_{p \in M} (T_pM)^*$
  - $S^*M := \left\{ v^* \in T^*M \mid |v^*|_g = 1 \right\}$

Brieskorn manifolds

- Links of singularities are contact manifolds.
  - $f : \mathbb{C}^n \to \mathbb{C}$ holomorphic function
  - Assume $f(0) = 0$
  - Assume 0 is an isolated singularity of f
  - Link of f at 0: $f^{-1}(0) \cap \mathbb{S}^{2n-1}_\varepsilon$

Definition

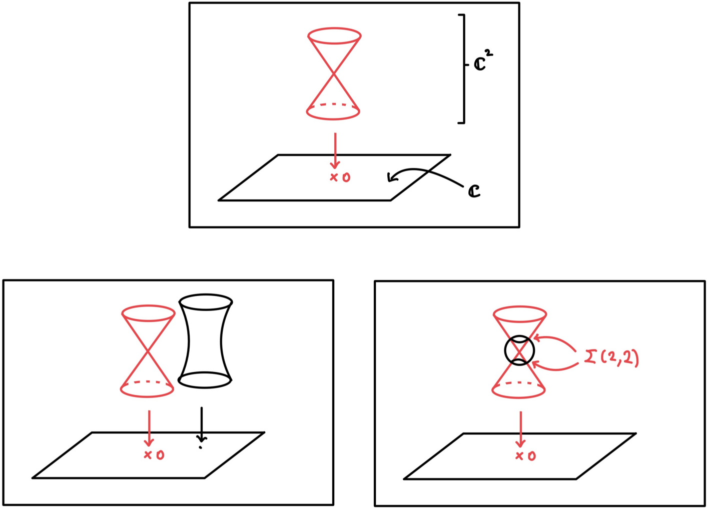
Let $a_0, \dots, a_n \in \mathbb{N}$. The Brieskorn manifold $\Sigma(a_0, \dots, a_n)$ is the link associated to the singularity 0 of the function

$$f : \mathbb{C}^{n+1} \to \mathbb{C} \quad : \quad (z_0, \dots, z_n) \mapsto z_0^{a_0} + \dots + z_n^{a_n}.$$

- Brieskorn 1966: $\Sigma(2, 2, 2, 3, 6k - 1)$ for $k = 1, 2, \dots, 28$ is a model for all possible smooth structures (up to a diffeomorphism) on $\mathbb{S}^7$.
- Ustilovsky 1999: If n is odd and $p = \pm 1 \pmod 8$, then $\Sigma(p, 2, \dots, 2)$ are not contactomorphic for different choices on p, but they are all diffeomorphic to the standard smooth sphere.

Brieskorn manifolds

- $\Sigma(2,2) \subset \mathbb{C}^2$



How big is Cont(M)?

- Let (M, ξ) be a closed contact manifold.
- Fix a contact form α on M. That is, pick α such that $\ker \alpha = \xi$.
- The contact form α determines a vector field R^α on M by the following relations: $\alpha(R^\alpha) = 1$ and $d\alpha(R^\alpha, \cdot) = 0$.
- The vector field R^α is called the *Reeb vector field*.

Fact

The flow $\phi_t^\alpha : M \to M$ of the Reeb vector field R^α consists of contactomorphisms.

- In particular, Cont(M, ξ) is "infinite-dimensional".
- More generally, every smooth function $h : M \times \mathbb{R} \to \mathbb{R}$ gives rise to a *contact isotopy*, i.e. a family of contactomorphisms, $\phi_t^h : M \to M$.
- The function h is called a *contact Hamiltonian*.

Contact Hamiltonians

Definition

Let M be a manifold with a contact form α. A contact Hamiltonian is a smooth function $h : M \to \mathbb{R}$. A time dependent contact Hamiltonian is a smooth function $h : M \times \mathbb{R} \to \mathbb{R}$.

- A (time-dependent) contact Hamiltonian h_t gives rise to a (time-dependent) vector field Y^{h_t} defined by
$$\alpha(Y^{h_t}) = -h_t, \quad d\alpha(Y^{h_t}, \cdot) = dh_t - dh(R^\alpha) \cdot \alpha.$$
- The flow of Y^{h_t} is a smooth family of contactomorphisms, denoted by $\phi_t^h : M \to M$.
- The Reeb flow is the contact isotopy of the contact Hamiltonian $h \equiv -1$.

Fact

Every contact isotopy is furnished by some contact Hamiltonian.

Contact non-squeezing

- There is no natural metric in contact geometry.
- There is no natural measure either.

Definition

Let (M, ξ) be a contact manifold. A subset $A \subset M$ can be contactly squeezed into a subset $B \subset M$ if there exists a compactly supported contact isotopy $\phi_t : M \to M$ such that $\phi_0 = \mathrm{id}$ and $\phi_1(A) \subset \mathrm{int}\, B$.

Non-squeezing and first examples

Fact

In the contact manifold $(\mathbb{R}^{2n+1}, \xi^\circ)$ every bounded subset $A \subset \mathbb{R}^{2n+1}$ can be contactly squeezed into any open non-empty $B \subset \mathbb{R}^{2n+1}$.

- Homotety-like contact isotopy $\phi_s \, : \, (x, y, z) \mapsto (s \cdot x, s \cdot y, s^2 \cdot z)$
- ϕ_s is generated by the contact Hamiltonian $h_s(x, y, z) = -\frac{2z}{s}$.

Fact

In the contact manifold $(\mathbb{S}^{2n+1}, \xi^{st})$ every non-dense subset $A \subset \mathbb{S}^{2n+1}$ can be contactly squeezed into any non-empty open subset $B \subset \mathbb{S}^{2n+1}$.

- Recall : $\mathbb{R}^{2n+1}$ and $\mathbb{S}^{2n+1} \setminus \{pt\}$ are contactomorphic.

Fact

There is no non-trivial contact non-squeezing on a small scale.

Non-squeezing on a large scale

- $B(R) =$ closed ball of radius R in $\mathbb{R}^{2n}$

Theorem (Eliashberg-Kim-Polterovich, Chiu)

$B(R) \times \mathbb{S}^1 \subset \mathbb{R}^{2n} \times \mathbb{S}^1$ *can be contactly squeezed into itself iff* $R < 1$.

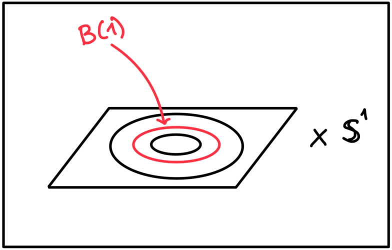
- Eliashberg-Kim-Polterovich : $R < 1$ or $R \in \mathbb{N}$
- Chiu: not necessarily integer R
- Alternative proofs: Fraser, Sandon

Fact

There is a non-trivial contact non-squeezing on a large scale.

- Smoothly, $B(R) \times \mathbb{S}^1$ can always be squeezed into itself.

Non-squeezing on a large scale



- Full tori inside the interior of $B(1) \times \mathbb{S}^1$ can be arbitrarily contactly squeezed.
- Full tori containing $B(1) \times \mathbb{S}^1$ cannot be contactly squeezed (not even a little bit).
- Why is 1 special?
- $\mathbb{S}^1$ is rescaled such that its length is 1.

Non-squeezing of a ball in a smooth sphere

- Ustilovsky spheres: $\Sigma(p, 2, \ldots, 2)$, $p \equiv \pm 1 \pmod 8$

$$\left\{ z_0^p + z_1^2 + \cdots z_{2m+1}^2 = 0 \, \& \, |z| = 1 \right\} \subset \mathbb{C}^{2m+2}$$

- Ustilovsky spheres are diffeomorphic to standard smooth spheres.
- Contact distribution on an Ustilovsky sphere is homotopic to the standard contact structure on the sphere if $p \equiv 1 \pmod{2 \cdot (2m)!}$.

Theorem (U.)

In every Ustilovsky sphere there exist two smoothly embedded closed balls B_1 and B_2 of maximal dimension such that B_2 cannot be contactly squeezed into B_1.

Non-squeezing on homotopy spheres

- Liouville domains = certain compact symplectic manifolds with boundary.
- The boundary of a Liouville domain is a contact manifold.
- Symplectic homology $SH_*(W)$ is an invariant of a Luouville domain W based on Floer theory.
- The Brieskorn manifold $\Sigma(a_0, \dots, a_n)$ is the boundary of a Liouville odmain $V(a_0, \dots, a_n)$, that very often has infinite-dimensional symplectic homology.

Theorem (U.)

Let W be a Liouville domain of dimension $2n \geqslant 4$ such that

- $\dim SH_*(W) = \infty$,
- *∂W is a homotopy sphere.*

Then, there exist two embedded closed balls $B_1, B_2 \subset \partial W$ of dimesion $2n - 1$ such that B_2 cannot be contactly squeezed into B_1.

Homotopy spheres and smooth squeezing

- Homotopy spheres are manifolds that are homotopy equivalent to spheres.
- Smale, Freedman, Perelman: homotopy spheres are homeomorphic to spheres.
- Smooth homotopy spheres, except perhaps in dimension 4, admit Morse functions with precisely 2 critical points.

Fact

Let M be a smooth homotopy sphere of dimension $\neq 4$. Then, every non-dense subset $A \subset M$ can be smoothly squeezed into any non-empty open subset $B \subset M$.

- Use the gradient flow of a Morse function with 2 critical points to do the squeezing.

Corollary

Contact non-squeezing on homotopy spheres is genuinely contact phenomenon.

Exotic contact $\mathbb{R}^{4m+1}$ and non-squeezing

Theorem (U.)

There exist a contact structure on $\mathbb{R}^{4m+1}$ and two embedded closed balls $B_1, B_2 \subset \mathbb{R}^{4m+1}$ such that B_2 cannot be contactly squeezed into B_1.

- Remove a point from an Ustilovsky sphere.

Corollary

The standard contact $\mathbb{R}^{4m+1}$ is not contactomorphic to any Ustilovsky sphere with a point removed.

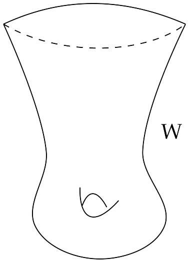
Theorem (Fauteux-Chapleau and Helfer)

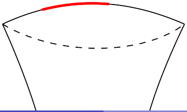
There exist infinitely many pairwise non-contactomorphic tight contact structures on $\mathbb{R}^{2n+1}$ if $n > 1$.

Selective symplectic homology

$$SH_*^{\Omega}(W)$$

- W a Liouville domain
- In particular, ∂W is a contact manifold.
- $\Omega \subset \partial W$ an open subset of the boundary





Selective symplectic homology

- For $\Omega_a \subset \Omega_b \subset \partial W$, there is a well-defined continuation map
$$\Phi_a^b : SH_*^{\Omega_a}(W) \to SH_*^{\Omega_b}(W).$$
- $\Phi_a^a = \mathrm{id}$
- $\Phi_b^c \circ \Phi_a^b = \Phi_a^c$

Fact

The graded vector spaces $SH_*^{\Omega}(W)$ together with the continuation maps Φ form a directed system indexed by open subsets of ∂W with order relation $\subset$.

Proposition

Let $\Omega_k \subset \partial W$ be an increasing sequence of open subsets. Denote $\Omega := \bigcup_k \Omega_k$.Then, the map
$$\varinjlim_k SH_*^{\Omega_k}(W) \to SH_*^{\Omega}(W)$$
furnished by continuation maps is an isomorphism.

Selective symplectic homology

Proposition

Let W be a Liouville domain. Let $\phi : \partial W \to \partial W$ be a contactomorphism that is (contactly) isotopic to the identity. Then, there exists an isomorphism

$$\mathcal{C}(\phi) : SH_*^{\Omega}(W) \to SH_*^{\phi(\Omega)}(W)$$

for every open subset $\Omega \subset \partial W$.

- $\mathcal{C}(\phi)$ depends actually on a little bit more than just ϕ. There are some additional choices involved.

Proposition

In the situation of the proposition above, if $\Omega_a \subset \Omega_b$ are open subsets of ∂W, then the following diagram commutes:

$$\begin{array}{ccc} SH_*^{\Omega_a}(W) & \xrightarrow[\mathcal{C}(\phi)]{} & SH_*^{\phi(\Omega_a)}(W) \\ {\scriptstyle \Phi_a^b}\downarrow & & \downarrow{\scriptstyle \Phi_a^b} \\ SH_*^{\Omega_b}(W) & \xrightarrow{\mathcal{C}(\phi)} & SH_*^{\phi(\Omega_b)}(W). \end{array}$$

Selective symplectic homology

- For $\Omega \subset \partial W$ open, denote
$$r(\Omega) := \operatorname{rank}\left(\Phi : SH_*^{\Omega}(W) \to SH_*^{\partial W}(W)\right).$$
- If Ω_a and Ω_b are contact isotopic, then $r(\Omega_a) = r(\Omega_b)$.

Theorem (U.)

If $r(\Omega_a) < r(\Omega_b)$, then Ω_b cannot be contactly squeezed into Ω_a.

- Compute $SH_*^{\Omega}(W)$!
- $SH_*^{\varnothing}(W) \cong H_{*+n}(W, \partial W; \mathbb{Z}_2)$
- $SH_*^{\partial W}(W) = SH_*(W)$

Selective symplectic homology

- Every contact manifold locally looks like $(\mathbb{R}^{2n+1}, \xi^\circ)$.
- Contact manifold can be covered by charts in which it looks like a subset of $(\mathbb{R}^{2n+1}, \xi^\circ)$.
- These charts are called contact Darboux charts.

Proposition

Let W be a Liouville domain and let $B \subset \partial W$ be a closed ball in a contact Darboux chart. Then, the continuation map

$$SH_*^{\varnothing}(W) \to SH_*^{\operatorname{int} B}(W)$$

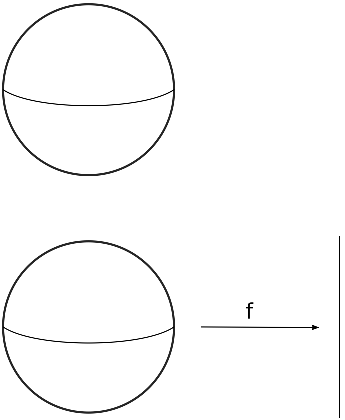
is an isomorphism.

Proposition

Let W be a Liouville domain with $\dim SH_*(W) = \infty$ and $\dim W \geqslant 4$. Then, for every $C \in \mathbb{R}$, there exists a contact Darboux chart D such that the continuation map $SH_*^{\partial W \setminus D}(W) \to SH_*(W)$ has rank greater than C.

Proof of the non-squeezing

- Choose Ω_S such that $r(\Omega_S) < \infty$.
- Choose Ω_N such that $r(\partial W \setminus \Omega_N) > r(\Omega_S)$.
- Recall $r(\Omega) = \mathrm{rk}\left(SH_*^{\Omega}(W) \to SH_*(W)\right)$.



Thank you!