Configuration spaces and the Johnson filtration

March 11, 2024 – Moduli and friends

Andreas Stavrou

University of Chicago

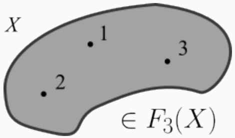
Configuration spaces

Configuration spaces

The n^{th} **ordered configuration space** of a space X is

$$F_n(X) := \{(x_1, ..., x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\},$$

topologised as a subset of X^n.

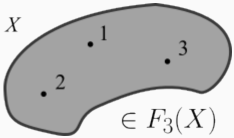


Configuration spaces

The n^{th} **ordered configuration space** of a space X is

$$F_n(X) := \{(x_1, ..., x_n) \in X^n : x_i \neq x_j \text{ for } i \neq j\},$$

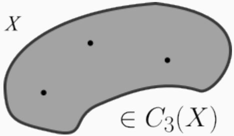
topologised as a subset of X^n.



The n^{th} **unordered configuration space** of a space X is the quotient

$$C_n(X) := F_n(X)/\mathfrak{S}_n,$$

by the permutation action of the symmetric group $\mathfrak{S}_n$.



In context

In context

- Braid groups are π_1 of $F_n(\mathbb{R}^2)$ and $C_n(\mathbb{R}^2)$.

In context

- Braid groups are π_1 of $F_n(\mathbb{R}^2)$ and $C_n(\mathbb{R}^2)$.
- Embedding calculus: study an embedding a few points at a time $\rightarrow$ knot invariants. (Teichner, Watanabe, ...)

In context

- Braid groups are π_1 of $F_n(\mathbb{R}^2)$ and $C_n(\mathbb{R}^2)$.
- Embedding calculus: study an embedding a few points at a time $\to$ knot invariants. (Teichner, Watanabe, ...)
- Homotopy equivalent but not homeomeomorphic lens spaces can be distinguished by their configuration spaces. ($L_{7,1} \simeq L_{7,2}$ but $L_{7,1} \not\cong L_{7,2}$; $F_2(L_{7,1}) \not\simeq F_2(L_{7,2})$.) (Longoni-Salvatore '04).

In context

- Braid groups are π_1 of $F_n(\mathbb{R}^2)$ and $C_n(\mathbb{R}^2)$.
- Embedding calculus: study an embedding a few points at a time $\to$ knot invariants. (Teichner, Watanabe, ...)
- Homotopy equivalent but not homeomeomorphic lens spaces can be distinguished by their configuration spaces. ($L_{7,1} \simeq L_{7,2}$ but $L_{7,1} \not\cong L_{7,2}$; $F_2(L_{7,1}) \not\simeq F_2(L_{7,2})$.) (Longoni-Salvatore '04).

Motto

The homology of a space loses information. The homology of configurations of the space retains more information.

Mapping class group actions

Mapping class group actions

- Let M be an oriented manifold, possibly with boundary. The **mapping class group** of M is the group

$$\begin{aligned}\mathrm{MCG}(M) &= \pi_0(\mathrm{Diff}^+_\partial(M)) \\ &= \text{oriented self-diffeos of } M\text{, fixing } \partial M, \text{ up to isotopy.}\end{aligned}$$

Mapping class group actions

- Let M be an oriented manifold, possibly with boundary. The **mapping class group** of M is the group

$$\begin{aligned}\mathrm{MCG}(M) &= \pi_0(\mathrm{Diff}^+_\partial(M)) \\ &= \text{oriented self-diffeos of } M\text{, fixing } \partial M,\ \text{up to isotopy.}\end{aligned}$$

- There is a natural action $\mathrm{MCG}(M) \curvearrowright H_*(M)$.

Mapping class group actions

- Let M be an oriented manifold, possibly with boundary. The **mapping class group** of M is the group

$$\begin{aligned} \mathrm{MCG}(M) &= \pi_0(\mathrm{Diff}^+_\partial(M)) \\ &= \text{oriented self-diffeos of } M\text{, fixing } \partial M, \text{ up to isotopy.} \end{aligned}$$

- There is a natural action $\mathrm{MCG}(M) \curvearrowright H_*(M)$.
- There is also an action

$$\mathrm{Diff}^+_\partial(M) \curvearrowright F_n(M), C_n(M)$$

which descends to

$$\mathrm{MCG}(M) \curvearrowright H_*(F_n(M)), H_*(C_n(M)).$$

Surfaces

Surfaces

- Let $\Sigma_{g,1}$ be the compact orientable genus g surface with one boundary component.

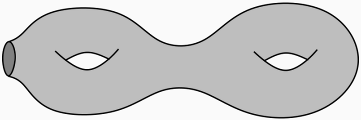


Figure 1: A $\Sigma_{2,1}$ surface.

Surfaces

- Let $\Sigma_{g,1}$ be the compact orientable genus g surface with one boundary component.

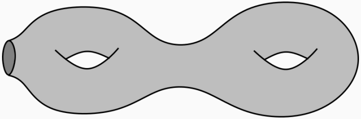


Figure 1: A $\Sigma_{2,1}$ surface.

- Write $\Gamma_{g,1} := \mathrm{MCG}(\Sigma_{g,1})$.

Surfaces

- Let $\Sigma_{g,1}$ be the compact orientable genus g surface with one boundary component.

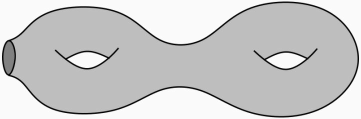


Figure 1: A $\Sigma_{2,1}$ surface.

- Write $\Gamma_{g,1} := \mathrm{MCG}(\Sigma_{g,1})$.
- The natural action

$$\Gamma_{g,1} \curvearrowright H_1(\Sigma_{g,1}) \cong \mathbb{Z}^{2g}$$

has kernel called the **Torelli group**, denoted $I_{g,1}$. This group is large and complicated.

Non-trivial Torelli actions

Non-trivial Torelli actions

Theorem (Bianchi '19)

If $g \geq 2$, then $I_{g,1}$ acts non-trivially on $H_2(C_2(\Sigma_{g,1}))$.

Non-trivial Torelli actions

Theorem (Bianchi '19)

If $g \geq 2$, then $I_{g,1}$ acts non-trivially on $H_2(C_2(\Sigma_{g,1}))$.

- Let Σ_g be the closed counterpart of $\Sigma_{g,1}$, $\Gamma_g = \mathrm{MCG}(\Sigma_g)$ and $I_g = \ker(\Gamma_g \curvearrowright H_1(\Sigma_g))$.

Non-trivial Torelli actions

Theorem (Bianchi '19)

If $g \geq 2$, then $I_{g,1}$ acts non-trivially on $H_2(C_2(\Sigma_{g,1}))$.

- Let Σ_g be the closed counterpart of $\Sigma_{g,1}$, $\Gamma_g = \mathrm{MCG}(\Sigma_g)$ and $I_g = \ker(\Gamma_g \curvearrowright H_1(\Sigma_g))$.

Theorem (Looijenga '20, S. '20)

If $g \geq 3$, then I_g acts non-trivially on $H_3(C_3(\Sigma_g))$.

Non-trivial Torelli actions

Theorem (Bianchi '19)

If $g \geq 2$, then $I_{g,1}$ acts non-trivially on $H_2(C_2(\Sigma_{g,1}))$.

- Let Σ_g be the closed counterpart of $\Sigma_{g,1}$, $\Gamma_g = \mathrm{MCG}(\Sigma_g)$ and $I_g = \ker(\Gamma_g \curvearrowright H_1(\Sigma_g))$.

Theorem (Looijenga '20, S. '20)

If $g \geq 3$, then I_g acts non-trivially on $H_3(C_3(\Sigma_g))$.

Question

The homology of configuration spaces of surfaces sees more of the mapping class group than the homology of the surface. How much more?

The action on unordered configurations

The action on unordered configurations

- For **unordered** configurations, the answer is: not that much.

The action on unordered configurations

- For **unordered** configurations, the answer is: not that much.
- The **Johnson kernel**, $K_{g,1}$, is the subgroup of $\Gamma_{g,1}$ generated by separating Dehn twists. In fact, $K_{g,1} \subset I_{g,1}$.

The action on unordered configurations

- For **unordered** configurations, the answer is: not that much.
- The **Johnson kernel**, $K_{g,1}$, is the subgroup of $\Gamma_{g,1}$ generated by separating Dehn twists. In fact, $K_{g,1} \subset I_{g,1}$.

Theorem (S. '21)

$K_{g,1}$ acts trivially on $H_*(C_n(\Sigma_{g,1}); \mathbb{Q})$, for all $n, g \geq 0$.

The action on unordered configurations

- For **unordered** configurations, the answer is: not that much.
- The **Johnson kernel**, $K_{g,1}$, is the subgroup of $\Gamma_{g,1}$ generated by separating Dehn twists. In fact, $K_{g,1} \subset I_{g,1}$.

Theorem (S. '21)

$K_{g,1}$ acts trivially on $H_*(C_n(\Sigma_{g,1});\mathbb{Q})$, for all $n, g \geq 0$.

Theorem (Bianchi–S. '23)

In fact, $K_{g,1} = \ker(\Gamma_{g,1} \curvearrowright H_*(C_n(\Sigma_{g,1})))$, for all $n, g \geq 2$.

The action on unordered configurations

- For **unordered** configurations, the answer is: not that much.
- The **Johnson kernel**, $K_{g,1}$, is the subgroup of $\Gamma_{g,1}$ generated by separating Dehn twists. In fact, $K_{g,1} \subset I_{g,1}$.

Theorem (S. '21)

$K_{g,1}$ acts trivially on $H_*(C_n(\Sigma_{g,1}); \mathbb{Q})$, for all $n, g \geq 0$.

Theorem (Bianchi–S. '23)

In fact, $K_{g,1} = \ker(\Gamma_{g,1} \curvearrowright H_*(C_n(\Sigma_{g,1}))$, for all $n, g \geq 2$.

Theorem (S. '23)

$K_g = \ker(\Gamma_g \curvearrowright H_*(C_n(\Sigma_g))$, for all $n, g \geq 3$.

The action on unordered configurations

- For **unordered** configurations, the answer is: not that much.
- The **Johnson kernel**, $K_{g,1}$, is the subgroup of $\Gamma_{g,1}$ generated by separating Dehn twists. In fact, $K_{g,1} \subset I_{g,1}$.

Theorem (S. '21)

$K_{g,1}$ acts trivially on $H_*(C_n(\Sigma_{g,1}); \mathbb{Q})$, for all $n, g \geq 0$.

Theorem (Bianchi–S. '23)

In fact, $K_{g,1} = \ker(\Gamma_{g,1} \curvearrowright H_*(C_n(\Sigma_{g,1}))$, for all $n, g \geq 2$.

Theorem (S. '23)

$K_g = \ker(\Gamma_g \curvearrowright H_*(C_n(\Sigma_g))$, for all $n, g \geq 3$.

- What about **ordered** configurations?

The Johnson filtration

The Johnson filtration

- The n^{th} **Johnson subgroup** of $\Gamma_{g,1}$ is the kernel

$$J(n) = \ker(\Gamma_{g,1} \curvearrowright \pi_1(\Sigma_{g,1})/\pi_1(\Sigma_{g,1})^{(n+1)}),$$

  where $G^{(i)}$ is the i^{th} commutator subgroup of G:

$$G^{(1)} = G, G^{(i+1)} = [G, G^{(i)}].$$

The Johnson filtration

- The n^{th} **Johnson subgroup** of $\Gamma_{g,1}$ is the kernel

$$J(n) = \ker(\Gamma_{g,1} \curvearrowright \pi_1(\Sigma_{g,1})/\pi_1(\Sigma_{g,1})^{(n+1)}),$$

  where $G^{(i)}$ is the i^{th} commutator subgroup of G:

$$G^{(1)} = G, G^{(i+1)} = [G, G^{(i)}].$$

- Notice $J(0) = \Gamma_{g,1}$, $J(1) = I_{g,1}$, $J(2) = K_{g,1}$.

The Johnson filtration

- The n^{th} **Johnson subgroup** of $\Gamma_{g,1}$ is the kernel

$$J(n) = \ker(\Gamma_{g,1} \curvearrowright \pi_1(\Sigma_{g,1})/\pi_1(\Sigma_{g,1})^{(n+1)}),$$

  where $G^{(i)}$ is the i^{th} commutator subgroup of G:

$$G^{(1)} = G, G^{(i+1)} = [G, G^{(i)}].$$

- Notice $J(0) = \Gamma_{g,1}$, $J(1) = I_{g,1}$, $J(2) = K_{g,1}$.
- We have the **Johnson filtration**

$$J(0) \supset J(1) \supset J(2) \supset J(3) \supset \ldots$$

  which is exhaustive: $\bigcap_{n \geq 0} J(n) = \{1\}$.

The Johnson filtration

- The n^{th} **Johnson subgroup** of $\Gamma_{g,1}$ is the kernel

$$J(n) = \ker(\Gamma_{g,1} \curvearrowright \pi_1(\Sigma_{g,1})/\pi_1(\Sigma_{g,1})^{(n+1)}),$$

  where $G^{(i)}$ is the i^{th} commutator subgroup of G:

$$G^{(1)} = G, G^{(i+1)} = [G, G^{(i)}].$$

- Notice $J(0) = \Gamma_{g,1}$, $J(1) = I_{g,1}$, $J(2) = K_{g,1}$.
- We have the **Johnson filtration**

$$J(0) \supset J(1) \supset J(2) \supset J(3) \supset \dots$$

  which is exhaustive: $\bigcap_{n \geq 0} J(n) = \{1\}$.

Theorem (Bianchi–Miller–Wilson '21)

The n^{th} Johnson subgroup $J(n)$ acts trivially on $H_*(F_n(\Sigma_{g,1}))$, for all $n \geq 1$.

Moriyama’s representations

Moriyama's representations

- Pick a point $p \in \partial\Sigma_{g,1}$ and define the subspaces of $(\Sigma_{g,1})^n$

$$\Delta_n = \{(x_1, ..., x_n) : x_i = x_j \text{ for } i \neq j\}$$

and

$$A_n = \{(x_1, ..., x_n) : x_i = p \text{ for some } i\}.$$

Moriyama's representations

- Pick a point $p \in \partial\Sigma_{g,1}$ and define the subspaces of $(\Sigma_{g,1})^n$

$$\Delta_n = \{(x_1, ..., x_n) : x_i = x_j \text{ for } i \neq j\}$$

  and

$$A_n = \{(x_1, ..., x_n) : x_i = p \text{ for some } i\}.$$

- The pair $((\Sigma_{g,1})^n, \Delta_n \cup A_n)$ is acted on naturally by $\mathrm{Diff}(\Sigma_{g,1})$ and has homology only in degree n.

Moriyama's representations

- Pick a point $p \in \partial\Sigma_{g,1}$ and define the subspaces of $(\Sigma_{g,1})^n$

$$\Delta_n = \{(x_1, \ldots, x_n) : x_i = x_j \text{ for } i \neq j\}$$

  and

$$A_n = \{(x_1, \ldots, x_n) : x_i = p \text{ for some } i\}.$$

- The pair $((\Sigma_{g,1})^n, \Delta_n \cup A_n)$ is acted on naturally by $\mathrm{Diff}(\Sigma_{g,1})$ and has homology only in degree n.

Theorem (Moriyama 2007)

The kernel of the $\Gamma_{g,1}$-representation $H_n((\Sigma_{g,1})^n, \Delta_n \cup A_n)$ is $J(n)$.

Sketch of proof Bianchi-Miller-Wilson

Sketch of proof Bianchi-Miller-Wilson

Theorem (Bianchi–Miller–Wilson '21)

The subgroup $J(n)$ acts trivially on $H_*(F_n(\Sigma_{g,1}))$.

Sketch of proof Bianchi-Miller-Wilson

Theorem (Bianchi–Miller–Wilson '21)

The subgroup $J(n)$ acts trivially on $H_*(F_n(\Sigma_{g,1}))$.

Sketch proof.

□

Sketch of proof Bianchi-Miller-Wilson

Theorem (Bianchi–Miller–Wilson ‘21)

The subgroup $J(n)$ acts trivially on $H_*(F_n(\Sigma_{g,1}))$.

Sketch proof.

- Define $B_n = \{(x_1, ..., x_n) : x_i \in \partial\Sigma_{g,1} \text{ for some } i\} \subset (\Sigma_{g,1})^n$.

□

Sketch of proof Bianchi-Miller-Wilson

Theorem (Bianchi–Miller–Wilson '21)

The subgroup $J(n)$ acts trivially on $H_*(F_n(\Sigma_{g,1}))$.

Sketch proof.

- Define $B_n = \{(x_1, ..., x_n) : x_i \in \partial\Sigma_{g,1} \text{ for some } i\} \subset (\Sigma_{g,1})^n$.
- By Poincaré-Lefschetz duality,

$$H^*(F_n(\Sigma_{g,1})) \cong H_{2n-*}((\Sigma_{g,1})^n, \Delta_n \cup B_n).$$

□

Sketch of proof Bianchi-Miller-Wilson

Theorem (Bianchi–Miller–Wilson '21)

The subgroup $J(n)$ acts trivially on $H_*(F_n(\Sigma_{g,1}))$.

Sketch proof.

- Define $B_n = \{(x_1, \ldots, x_n) : x_i \in \partial\Sigma_{g,1} \text{ for some } i\} \subset (\Sigma_{g,1})^n$.
- By Poincaré-Lefschetz duality,

$$H^*(F_n(\Sigma_{g,1})) \cong H_{2n-*}((\Sigma_{g,1})^n, \Delta_n \cup B_n).$$

- Construct a chain-complex, natural with the $\Gamma_{g,1}$-action, which computes $\oplus_{n\geq 0} H_*((\Sigma_{g,1})^n, \Delta_n \cup B_n)$.

□

Sketch of proof Bianchi-Miller-Wilson

Theorem (Bianchi–Miller–Wilson '21)

The subgroup $J(n)$ acts trivially on $H_*(F_n(\Sigma_{g,1}))$.

Sketch proof.

- Define $B_n = \{(x_1, \dots, x_n) : x_i \in \partial\Sigma_{g,1} \text{ for some } i\} \subset (\Sigma_{g,1})^n$.
- By Poincaré-Lefschetz duality,

$$H^*(F_n(\Sigma_{g,1})) \cong H_{2n-*}((\Sigma_{g,1})^n, \Delta_n \cup B_n).$$

- Construct a chain-complex, natural with the $\Gamma_{g,1}$-action, which computes $\oplus_{n\geq 0} H_*((\Sigma_{g,1})^n, \Delta_n \cup B_n)$.
- The part that computes $H_*((\Sigma_{g,1})^n, \Delta_n \cup B_n)$ involves, before taking homology, only $H_k((\Sigma_{g,1})^k, \Delta_k \cup A_k)$ for $k \leq n$.

□

Sketch of proof Bianchi-Miller-Wilson

Theorem (Bianchi–Miller–Wilson '21)

The subgroup $J(n)$ acts trivially on $H_*(F_n(\Sigma_{g,1}))$.

Sketch proof.

- Define $B_n = \{(x_1, ..., x_n) : x_i \in \partial\Sigma_{g,1}$ for some $i\} \subset (\Sigma_{g,1})^n$.
- By Poincaré-Lefschetz duality,

$$H^*(F_n(\Sigma_{g,1})) \cong H_{2n-*}((\Sigma_{g,1})^n, \Delta_n \cup B_n).$$

- Construct a chain-complex, natural with the $\Gamma_{g,1}$-action, which computes $\oplus_{n\geq 0} H_*((\Sigma_{g,1})^n, \Delta_n \cup B_n)$.
- The part that computes $H_*((\Sigma_{g,1})^n, \Delta_n \cup B_n)$ involves, before taking homology, only $H_k((\Sigma_{g,1})^k, \Delta_k \cup A_k)$ for $k \leq n$.
- Therefore, $J(n)$ acts trivially. (But after taking homology the kernel might be bigger.)

□

Group Rings and the augmentation ideal

Group Rings and the augmentation ideal

- Let's motivate how $J(n)$ appear in Moriyama's theorem.

Group Rings and the augmentation ideal

- Let's motivate how $J(n)$ appear in Moriyama's theorem.
- Write $\pi = \pi_1(\Sigma_{g,1})$, a free group of rank $2g$. Consider the group ring $\mathbb{Z}[\pi]$.

Group Rings and the augmentation ideal

- Let's motivate how $J(n)$ appear in Moriyama's theorem.
- Write $\pi = \pi_1(\Sigma_{g,1})$, a free group of rank $2g$. Consider the group ring $\mathbb{Z}[\pi]$.
- Define the augmentation ideal $I \subset \mathbb{Z}[\pi]$ as the kernel of the augmentation map

$$\varepsilon : \mathbb{Z}[\pi] \to \mathbb{Z}$$
$$\gamma \in \pi \to 1.$$

Group Rings and the augmentation ideal

- Let's motivate how $J(n)$ appear in Moriyama's theorem.
- Write $\pi = \pi_1(\Sigma_{g,1})$, a free group of rank $2g$. Consider the group ring $\mathbb{Z}[\pi]$.
- Define the augmentation ideal $I \subset \mathbb{Z}[\pi]$ as the kernel of the augmentation map

$$\varepsilon : \mathbb{Z}[\pi] \to \mathbb{Z}$$
$$\gamma \in \pi \to 1.$$

- It is the two-sided ideal generated by $\gamma - 1$ for $\gamma \in \pi$.

Group Rings and the augmentation ideal

- Let's motivate how $J(n)$ appear in Moriyama's theorem.
- Write $\pi = \pi_1(\Sigma_{g,1})$, a free group of rank $2g$. Consider the group ring $\mathbb{Z}[\pi]$.
- Define the augmentation ideal $I \subset \mathbb{Z}[\pi]$ as the kernel of the augmentation map

$$\varepsilon : \mathbb{Z}[\pi] \to \mathbb{Z}$$
$$\gamma \in \pi \to 1.$$

- It is the two-sided ideal generated by $\gamma - 1$ for $\gamma \in \pi$.
- The powers I^n are also two-sided ideals generated by $(\gamma_1 - 1)(\gamma_2 - 1)...(\gamma_n - 1)$ for $\gamma_i \in \pi$.

Example calculation

Example calculation

- Example: check that if $\gamma \in \pi$ is a commutator then $\gamma - 1 \in I^2$.

Example calculation

- Example: check that if $\gamma \in \pi$ is a commutator then $\gamma - 1 \in I^2$.
- Write $\gamma = [\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$, then

$$\begin{aligned}\gamma - 1 &= \alpha\beta\alpha^{-1}\beta^{-1} - 1\\ &= \alpha\beta\alpha^{-1}\beta^{-1} - \beta\alpha\alpha^{-1}\beta^{-1}\\ &= (\alpha\beta - \beta\alpha)\alpha^{-1}\beta^{-1}\\ &= ((\alpha - 1)(\beta - 1) - (\beta - 1)(\alpha - 1))\alpha^{-1}\beta^{-1}.\end{aligned}$$

Example calculation

- Example: check that if $\gamma \in \pi$ is a commutator then $\gamma - 1 \in I^2$.
- Write $\gamma = [\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$, then

$$\begin{aligned}
\gamma - 1 &= \alpha\beta\alpha^{-1}\beta^{-1} - 1 \\
&= \alpha\beta\alpha^{-1}\beta^{-1} - \beta\alpha\alpha^{-1}\beta^{-1} \\
&= (\alpha\beta - \beta\alpha)\alpha^{-1}\beta^{-1} \\
&= ((\alpha - 1)(\beta - 1) - (\beta - 1)(\alpha - 1))\alpha^{-1}\beta^{-1}.
\end{aligned}$$

- $(\alpha - 1)(\beta - 1), (\beta - 1)(\alpha - 1) \in I^2$, thus $\gamma - 1 \in I^2$.

Example calculation

- Example: check that if $\gamma \in \pi$ is a commutator then $\gamma - 1 \in I^2$.
- Write $\gamma = [\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$, then

$$\begin{aligned} \gamma - 1 &= \alpha\beta\alpha^{-1}\beta^{-1} - 1 \\ &= \alpha\beta\alpha^{-1}\beta^{-1} - \beta\alpha\alpha^{-1}\beta^{-1} \\ &= (\alpha\beta - \beta\alpha)\alpha^{-1}\beta^{-1} \\ &= ((\alpha - 1)(\beta - 1) - (\beta - 1)(\alpha - 1))\alpha^{-1}\beta^{-1}. \end{aligned}$$

- $(\alpha - 1)(\beta - 1), (\beta - 1)(\alpha - 1) \in I^2$, thus $\gamma - 1 \in I^2$.

Theorem (Fox)

If $\gamma \in \pi$, then $\gamma - 1 \in I^n$ if and only if $\gamma \in \pi^{(n)}$, the n^{th} commutator subgroup of π.

Points travelling on loops

Points travelling on loops

- Let $\gamma : ([0,1], \partial\{0,1\}) \to (\Sigma_{g,1}, p)$ represent an element of π.

Points travelling on loops

- Let $\gamma : ([0,1], \partial\{0,1\}) \to (\Sigma_{g,1}, p)$ represent an element of π.
- Consider the n-simplex

$$\Delta^n = \{(t_1, ..., t_n) \in [0,1]^n : t_1 \leq t_2 \leq ... \leq t_n\}.$$

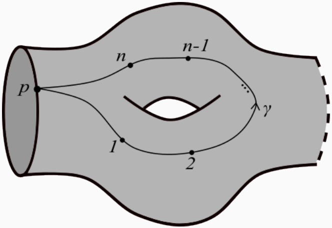
Points travelling on loops

- Let $\gamma : ([0,1], \partial\{0,1\}) \to (\Sigma_{g,1}, p)$ represent an element of π.
- Consider the n-simplex

$$\Delta^n = \{(t_1, ..., t_n) \in [0,1]^n : t_1 \leq t_2 \leq ... \leq t_n\}.$$

- Get map

$$\begin{aligned} f_n : \mathbb{Z}[\pi] &\to H_n((\Sigma_{g,1})^n, \Delta_n \cup A_n) \\ \gamma &\mapsto [\gamma^n | \Delta^n]. \end{aligned}$$



Moriyama: proof idea

Moriyama: proof idea

- The simultaneous kernel of $f_1, ..., f_n$ is I^{n+1}.

Moriyama: proof idea

- The simultaneous kernel of $f_1, \ldots, f_n$ is I^{n+1}.
- The images of $f_1, \ldots, f_n$ generate

$$\bigoplus_{i=1}^{n} H_i((\Sigma_{g,1})^i, \Delta_i \cup A_i).$$

Moriyama: proof idea

- The simultaneous kernel of $f_1, \dots, f_n$ is I^{n+1}.
- The images of $f_1, \dots, f_n$ generate

$$\bigoplus_{i=1}^{n} H_i((\Sigma_{g,1})^i, \Delta_i \cup A_i).$$

- It follows that the kernel of the $\Gamma_{g,1}$-representation $H_n((\Sigma_{g,1})^n, \Delta_n \cup A_n)$ is the same as the kernel of $\mathbb{Z}[\pi]/I^{n+1}$.

Moriyama: proof idea

- The simultaneous kernel of $f_1, \ldots, f_n$ is I^{n+1}.
- The images of $f_1, \ldots, f_n$ generate

$$\bigoplus_{i=1}^{n} H_i((\Sigma_{g,1})^i, \Delta_i \cup A_i).$$

- It follows that the kernel of the $\Gamma_{g,1}$-representation $H_n((\Sigma_{g,1})^n, \Delta_n \cup A_n)$ is the same as the kernel of $\mathbb{Z}[\pi]/I^{n+1}$.
- By Fox's theorem this is the same as the kernel of $\pi/\pi^{(n+1)}$.

Moriyama: proof idea

- The simultaneous kernel of $f_1, \ldots, f_n$ is I^{n+1}.
- The images of $f_1, \ldots, f_n$ generate

$$\bigoplus_{i=1}^{n} H_i((\Sigma_{g,1})^i, \Delta_i \cup A_i).$$

- It follows that the kernel of the $\Gamma_{g,1}$-representation $H_n((\Sigma_{g,1})^n, \Delta_n \cup A_n)$ is the same as the kernel of $\mathbb{Z}[\pi]/I^{n+1}$.
- By Fox's theorem this is the same as the kernel of $\pi/\pi^{(n+1)}$.
- This by definition is $J(n)$.

A conjecture

A conjecture

Conjecture (Bianchi–Miller–Wilson '21)

The kernel

$$\ker(\Gamma_{g,1} \curvearrowright H_*(F_n(\Sigma_{g,1})))$$

is the subgroup of $\Gamma_{g,1}$ generated by $J(n)$ and the Dehn twist $D_{\partial\Sigma_{g,1}}$ along $\partial\Sigma_{g,1}$.

A conjecture

Conjecture (Bianchi–Miller–Wilson '21)

The kernel

$$\ker(\Gamma_{g,1} \curvearrowright H_*(F_n(\Sigma_{g,1})))$$

is the subgroup of $\Gamma_{g,1}$ generated by $J(n)$ and the Dehn twist $D_{\partial\Sigma_{g,1}}$ along $\partial\Sigma_{g,1}$.

Theorem (Bianchi–S. '22)

For $n, g \geq 2$, the subgroup $J(n-1)$ acts non-trivially on $H_n(F_n(\Sigma_{g,1}))$.

- Idea: Construct $x \in H_n(F_n(\Sigma_{g,1}))$, $\phi \in J(n-1)$ and a reason why $\phi * x \neq x$.

A conjecture

Conjecture (Bianchi–Miller–Wilson '21)

The kernel

$$\ker(\Gamma_{g,1} \curvearrowright H_*(F_n(\Sigma_{g,1})))$$

is the subgroup of $\Gamma_{g,1}$ generated by $J(n)$ and the Dehn twist $D_{\partial\Sigma_{g,1}}$ along $\partial\Sigma_{g,1}$.

Theorem (Bianchi–S. '22)

For $n, g \geq 2$, the subgroup $J(n-1)$ acts non-trivially on $H_n(F_n(\Sigma_{g,1}))$.

- Idea: Construct $x \in H_n(F_n(\Sigma_{g,1}))$, $\phi \in J(n-1)$ and a reason why $\phi * x \neq x$.
- The class x comes from a closed submanifold $X \subset F_n(\Sigma_{g,1})$ of half-dimension.

A conjecture

Conjecture (Bianchi–Miller–Wilson '21)

The kernel

$$\ker(\Gamma_{g,1} \curvearrowright H_*(F_n(\Sigma_{g,1})))$$

is the subgroup of $\Gamma_{g,1}$ *generated by* $J(n)$ *and the Dehn twist* $D_{\partial\Sigma_{g,1}}$ *along* $\partial\Sigma_{g,1}$.

Theorem (Bianchi–S. '22)

For $n, g \geq 2$, *the subgroup* $J(n-1)$ *acts non-trivially on* $H_n(F_n(\Sigma_{g,1}))$.

- Idea: Construct $x \in H_n(F_n(\Sigma_{g,1}))$, $\phi \in J(n-1)$ and a reason why $\phi * x \neq x$.
- The class x comes from a closed submanifold $X \subset F_n(\Sigma_{g,1})$ of half-dimension.
- The "reason why" is a properly embedded submanifold $Y \subset F_n(\Sigma_{g,1})$ of half co-dimension that is disjoint from X, but intersects X transverally at finitely many points, so that the algebraic intersection number of X and Y is ± 1.

Foggy Roller Coasters

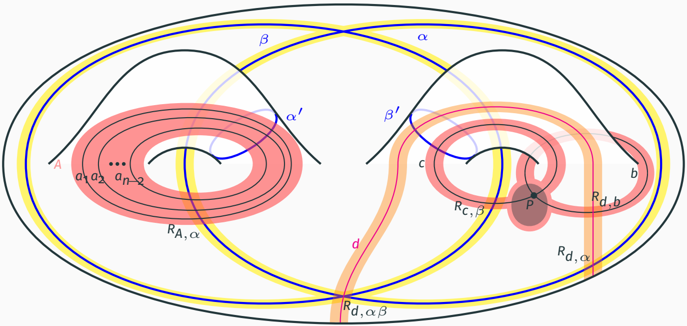


Figure 2: (A) The submanifold $X \subset (\Sigma_{g,1})^n$ is the n-torus supported in the pink region, with i^{th} configuration point orbiting on the parallel curves α_i if $i = 1, \ldots, n-2$, the $(n-1)^{th}$ point on curve b and n^{th} point on curve c.(B) The mapping class $\phi \in J(n-1)$ is the commutator of the Dehn twist along α repeated $(n-2)$ times and the Dehn twist along β. (C) The submanifold $Y \subset F_n(\Sigma_{g,1})$ corresponds to points 1, ..., n travelling along d in strictly increasing order. (D) The intersection of $\phi * X$ and Y takes place in the yellow neighbourhood of α and β. Ignoring what happens out of it can be thought of as *putting fog*.

Work in progress

Work in progress

- Let $\bar{\pi} = \pi_1(\Sigma_g)$. The marked mapping class group $\Gamma_{g,*}$ acts on $\bar{\pi}$ and can define a Johnson filtration $J_{g,*}(n)$ as the kernels on $\bar{\pi}/\bar{\pi}^{(n+1)}$.

Work in progress

- Let $\bar{\pi} = \pi_1(\Sigma_g)$. The marked mapping class group $\Gamma_{g,*}$ acts on $\bar{\pi}$ and can define a Johnson filtration $J_{g,*}(n)$ as the kernels on $\bar{\pi}/\bar{\pi}^{(n+1)}$.
- The map $\Gamma_{g,1} \to \Gamma_{g,*}$ is surjective and has kernel generated by $D_{\partial\Sigma_{g,1}}$.

Work in progress

- Let $\bar{\pi} = \pi_1(\Sigma_g)$. The marked mapping class group $\Gamma_{g,*}$ acts on $\bar{\pi}$ and can define a Johnson filtration $J_{g,*}(n)$ as the kernels on $\bar{\pi}/\bar{\pi}^{(n+1)}$.
- The map $\Gamma_{g,1} \to \Gamma_{g,*}$ is surjective and has kernel generated by $D_{\partial\Sigma_{g,1}}$.
- We have $F_n(\Sigma_{g,1}) \simeq F_n(\mathring{\Sigma}_{g,1}) \cong F_n(\Sigma_{g,*})$ in a Diff-equivariant way.

Work in progress

- Let $\bar{\pi} = \pi_1(\Sigma_g)$. The marked mapping class group $\Gamma_{g,*}$ acts on $\bar{\pi}$ and can define a Johnson filtration $J_{g,*}(n)$ as the kernels on $\bar{\pi}/\bar{\pi}^{(n+1)}$.
- The map $\Gamma_{g,1} \to \Gamma_{g,*}$ is surjective and has kernel generated by $D_{\partial\Sigma_{g,1}}$.
- We have $F_n(\Sigma_{g,1}) \simeq F_n(\mathring{\Sigma}_{g,1}) \cong F_n(\Sigma_{g,*})$ in a Diff-equivariant way.

Conjecture

The kernel of the $\Gamma_{g,*}$-action on $H_*(F_n(\Sigma_{g,*}))$ is $J_{g,*}(n)$.

Work in progress

- Let $\bar{\pi} = \pi_1(\Sigma_g)$. The marked mapping class group $\Gamma_{g,*}$ acts on $\bar{\pi}$ and can define a Johnson filtration $J_{g,*}(n)$ as the kernels on $\bar{\pi}/\bar{\pi}^{(n+1)}$.
- The map $\Gamma_{g,1} \to \Gamma_{g,*}$ is surjective and has kernel generated by $D_{\partial\Sigma_{g,1}}$.
- We have $F_n(\Sigma_{g,1}) \simeq F_n(\mathring{\Sigma}_{g,1}) \cong F_n(\Sigma_{g,*})$ in a Diff-equivariant way.

Conjecture

The kernel of the $\Gamma_{g,*}$-action on $H_*(F_n(\Sigma_{g,*}))$ is $J_{g,*}(n)$.

- In progress: conjecture true for $n = 2, 3$ by replicating Moriyama's work but for Σ_g.

Towards Johnson filtrations for all manifolds

Towards Johnson filtrations for all manifolds

- Simply connected manifolds (in higher dimensions) still have complicated mapping class groups, but no natural filtration that allows studying them at increasing depth.

Towards Johnson filtrations for all manifolds

- Simply connected manifolds (in higher dimensions) still have complicated mapping class groups, but no natural filtration that allows studying them at increasing depth.
- Perhaps we ought to **define** their Johnson filtration as

$$J_M(n) := \ker(\mathrm{MCG}(M) \curvearrowright H_*(F_n(M))).$$

Towards Johnson filtrations for all manifolds

- Simply connected manifolds (in higher dimensions) still have complicated mapping class groups, but no natural filtration that allows studying them at increasing depth.
- Perhaps we ought to **define** their Johnson filtration as

$$J_M(n) := \ker(\mathrm{MCG}(M) \curvearrowright H_*(F_n(M))).$$

- Proving the conjecture of Bianchi-Miller-Wilson would be an indication that this is a good idea.

Another story: Scanning

McDuff’s scanning map

McDuff's scanning map

- Let $\tau^{+}M$ be the fibrewise one-point compactification of the tangent bundle of M.

McDuff’s scanning map

- Let $\tau^+ M$ be the fibrewise one-point compactification of the tangent bundle of M.
- Let $\Gamma(M)$ be the space of sections of $\tau^+ M$ which take the value ∞ on ∂M.

McDuff's scanning map

- Let τ^+M be the fibrewise one-point compactification of the tangent bundle of M.
- Let $\Gamma(M)$ be the space of sections of τ^+M which take the value ∞ on ∂M.
- There is a **scanning** map $\sigma_n : C_n(M) \to \Gamma(M)$, which lands in the "degree-n" component $\Gamma_n(M)$ of $\Gamma(M)$.

McDuff's scanning map

- Let $\tau^+ M$ be the fibrewise one-point compactification of the tangent bundle of M.
- Let $\Gamma(M)$ be the space of sections of $\tau^+ M$ which take the value ∞ on ∂M.
- There is a **scanning** map $\sigma_n : C_n(M) \to \Gamma(M)$, which lands in the "degree-n" component $\Gamma_n(M)$ of $\Gamma(M)$.

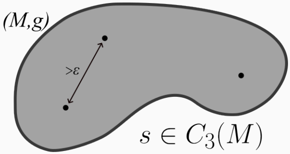
Theorem (McDuff '75, Randal-Williams '13)

The map $\sigma_n : C_n(M) \to \Gamma_n(M)$ induces an integral homology isomorphism in homological degrees $* \leq \frac{n}{2}$.

A definition of the scanning map

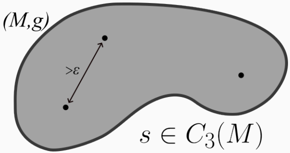
A definition of the scanning map

Fix a metric g on M. For every configuration $s \in C_n(M)$ there is an $\varepsilon > 0$ such that all points of s are at least ε apart.



A definition of the scanning map

Fix a metric g on M. For every configuration $s \in C_n(M)$ there is an $\varepsilon > 0$ such that all points of s are at least ε apart.

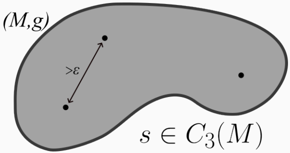


- Define $\sigma_n(s)$ to be the following section of T^+M:

$$x \in M \longmapsto \begin{cases} \infty \in T_x^+M, & \text{if there is no config point within } \varepsilon \text{ of } x, \\ \exp_{x,\varepsilon}^{-1}(p) \in T_xM, & \text{if } p \text{ is a config point within } \varepsilon \text{ of } x. \end{cases}$$

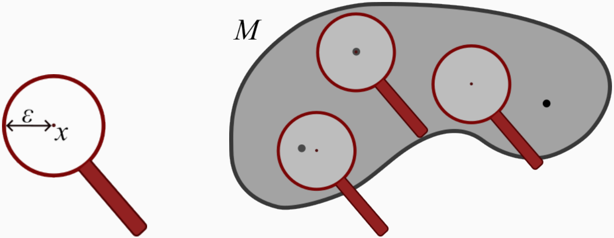
A definition of the scanning map

Fix a metric g on M. For every configuration $s \in C_n(M)$ there is an $\varepsilon > 0$ such that all points of s are at least ε apart.



- Define $\sigma_n(s)$ to be the following section of T^+M:

$$x \in M \longmapsto \begin{cases} \infty \in T_x^+M, & \text{if there is no config point within } \varepsilon \text{ of } x, \\ \exp_{x,\varepsilon}^{-1}(p) \in T_xM, & \text{if } p \text{ is a config point within } \varepsilon \text{ of } x. \end{cases}$$



Upgraded scanning

Upgraded scanning

- Let $T^{+}M \wedge_f S^m$ be the fibrewise one-point compactification of the tangent bundle of M fibrewise smashed with the sphere S^m. Let $\Gamma(M, S^m)$ be the the space of sections of $T^{+}M \wedge_f S^m$ supported in the interior of M.

Upgraded scanning

- Let $T^+M \wedge_f S^m$ be the fibrewise one-point compactification of the tangent bundle of M fibrewise smashed with the sphere S^m. Let $\Gamma(M, S^m)$ be the the space of sections of $T^+M \wedge_f S^m$ supported in the interior of M.

Theorem (Bödigheimer–Cohen–Taylor '80s)

For any $k \geq 1$, there is an isomorphism of vector spaces

$$\bigoplus_{n \geq 0} H^*(C_n(M))[2kn] \cong H^*(\Gamma(M, S^{2k})).$$

Upgraded scanning

- Let $T^+M \wedge_f S^m$ be the fibrewise one-point compactification of the tangent bundle of M fibrewise smashed with the sphere S^m. Let $\Gamma(M, S^m)$ be the the space of sections of $T^+M \wedge_f S^m$ supported in the interior of M.

Theorem (Bödigheimer–Cohen–Taylor '80s)

For any $k \geq 1$, there is an isomorphism of vector spaces

$$\bigoplus_{n \geq 0} H^*(C_n(M))[2kn] \cong H^*(\Gamma(M, S^{2k})).$$

Questions:

1. How to make the RHS more computable?
2. Both LHS and RHS have an action of MCG(M). Is this isomorphism MCG(M)-equivariant?
3. The RHS is a ring with the cup-product. Is this meaningful on the LHS?

Knudsen's theorem

Knudsen's theorem

Theorem (Knudsen '17)

There is an isomorphism of bigraded spaces

$$\bigoplus_{n\geq 0} H_*(C_n(M);\mathbb{Q}) \cong H^{\mathcal{L}}(H_c^{-*}(M;\mathcal{L}(\mathbb{Q}^w[d-1])))$$

where $d = \dim M$, $H^{\mathcal{L}}$ is Lie-algebra homology, $H_c^{-*}(M;\mathcal{L}(\mathbb{Q}^w[d-1]))$ is the compactly supported cohomology of M with coefficients in the free graded Lie algebra generated by the orientation sheaf $\mathbb{Q}^w$ of M in degree $d-1$.

Knudsen's theorem

Theorem (Knudsen '17)

There is an isomorphism of bigraded spaces

$$\bigoplus_{n\geq 0} H_*(C_n(M);\mathbb{Q}) \cong H^{\mathcal{L}}(H_c^{-*}(M;\mathcal{L}(\mathbb{Q}^w[d-1])))$$

where $d = \dim M$, $H^{\mathcal{L}}$ *is Lie-algebra homology,* $H_c^{-*}(M;\mathcal{L}(\mathbb{Q}^w[d-1]))$ *is the compactly supported cohomology of* M *with coefficients in the free graded Lie algebra generated by the orientation sheaf* $\mathbb{Q}^w$ *of* M *in degree* $d-1$*.*

- Proof uses factorization homology.

Knudsen's theorem

Theorem (Knudsen '17)

There is an isomorphism of bigraded spaces

$$\bigoplus_{n\geq 0} H_*(C_n(M);\mathbb{Q}) \cong H^{\mathcal{L}}(H_c^{-*}(M;\mathcal{L}(\mathbb{Q}^w[d-1])))$$

where $d = \dim M$, $H^{\mathcal{L}}$ *is Lie-algebra homology,* $H_c^{-*}(M;\mathcal{L}(\mathbb{Q}^w[d-1]))$ *is the compactly supported cohomology of* M *with coefficients in the free graded Lie algebra generated by the orientation sheaf* $\mathbb{Q}^w$ *of* M *in degree* $d-1$.

- Proof uses factorization homology.
- Computational power: used in Drummond-Cole–Knudsen '17 to compute the Betti numbers of $C_n(\Sigma)$ for all compact surfaces Σ.

Equivariant untwisted rational scanning

Equivariant untwisted rational scanning

Theorem (S. '21)

Let M be a compact, connected, oriented manifold of dimension d, and $k \geq 1$ an arbitrary integer. There is a $\mathrm{MCG}(M)$-equivariant isomorphism of graded vector spaces

$$\bigoplus_{i,n\geq 0} H^i(C_n(M);\mathbb{Q})[2kn] \cong H^*(\mathrm{map}_\partial(M, S^{d+2k});\mathbb{Q}).$$

Equivariant untwisted rational scanning

Theorem (S. '21)

Let M be a compact, connected, oriented manifold of dimension d, and $k \geq 1$ an arbitrary integer. There is a $\mathrm{MCG}(M)$-equivariant isomorphism of graded vector spaces

$$\bigoplus_{i,n \geq 0} H^i(C_n(M);\mathbb{Q})[2kn] \cong H^*(\mathrm{map}_\partial(M, S^{d+2k});\mathbb{Q}).$$

- Mapping spaces are easier than section spaces.

Equivariant untwisted rational scanning

Theorem (S. '21)

Let M be a compact, connected, oriented manifold of dimension d, and $k \geq 1$ an arbitrary integer. There is a $\mathrm{MCG}(M)$-equivariant isomorphism of graded vector spaces

$$\bigoplus_{i,n \geq 0} H^i(C_n(M);\mathbb{Q})[2kn] \cong H^*(\mathrm{map}_\partial(M, S^{d+2k});\mathbb{Q}).$$

- Mapping spaces are easier than section spaces.
- A second bidegree can be put on the RHS using the weights of the $\mathbb{Z}^\times$-action of automorphisms of the sphere, which makes this a bigraded isomorphism.

Equivariant untwisted rational scanning

Theorem (S. '21)

Let M be a compact, connected, oriented manifold of dimension d, and $k \geq 1$ an arbitrary integer. There is a $\mathrm{MCG}(M)$*-equivariant isomorphism of graded vector spaces*

$$\bigoplus_{i,n \geq 0} H^i(C_n(M);\mathbb{Q})[2kn] \cong H^*(\mathrm{map}_\partial(M, S^{d+2k});\mathbb{Q}).$$

- Mapping spaces are easier than section spaces.
- A second bidegree can be put on the RHS using the weights of the $\mathbb{Z}^\times$-action of automorphisms of the sphere, which makes this a bigraded isomorphism.
- Applied to compute $H^*(C_n(\Sigma_{g,1});\mathbb{Q})$ together with the $\Gamma_{g,1}$ action.

Superposition of configurations

Superposition of configurations

- For any compact manifold M with boundary $C_n(M) \simeq C_n(\mathring{M})$. In the **one-point compactification** $C_n(\mathring{M})^+$ you go to the "∞-configuration" if either:
    - two configuration points are colliding, or
    - one configuration point goes to ∂M.

Superposition of configurations

- For any compact manifold M with boundary $C_n(M) \simeq C_n(\mathring{M})$. In the **one-point compactification** $C_n(\mathring{M})^+$ you go to the "∞-configuration" if either:
  - two configuration points are colliding, or
  - one configuration point goes to ∂M.
- Define the **superposition** map

$$\begin{aligned} \sup_{n,m} : C_n(\mathring{M})^+ \times C_m(\mathring{M})^+ &\longrightarrow C_{n+m}(\mathring{M})^+ \\ (s,t) &\longmapsto \begin{cases} s \cup t, \text{ if } s,t \neq \infty \text{ and } s \cap t = \varnothing \\ \infty, \text{ otherwise.} \end{cases} \end{aligned}$$

Superposition of configurations

- For any compact manifold M with boundary $C_n(M) \simeq C_n(\mathring{M})$. In the **one-point compactification** $C_n(\mathring{M})^+$ you go to the "∞-configuration" if either:
  - two configuration points are colliding, or
  - one configuration point goes to ∂M.
- Define the **superposition** map

$$\sup_{n,m} : C_n(\mathring{M})^+ \times C_m(\mathring{M})^+ \longrightarrow C_{n+m}(\mathring{M})^+$$

$$(s,t) \longmapsto \begin{cases} s \cup t, \text{ if } s,t \neq \infty \text{ and } s \cap t = \varnothing \\ \infty, \text{ otherwise.} \end{cases}$$

- Using Poincaré-Lefschetz duality, we obtain the map $\mu_{m,n}$:

$$\begin{array}{ccc} H^i(C_n(M)) \otimes H^j(C_m(M)) & \overset{\mu_{m,n}}{\dashrightarrow} & H^{i+j}(C_{n+m}(M)) \\ \cong \downarrow PD & & \cong \downarrow PD \\ H_{dn-i}(C_n(\mathring{M})^+) \otimes H_{dm-j}(C_n(\mathring{M})^+) & \overset{\sup_{n,m*}}{\longrightarrow} & H_{d(n+m)-(i+j)}(C_{n+m}(\mathring{M})^+) \end{array}$$

The superposition product

The superposition product

- The maps $\mu_{m,n}$, for $m, n \geq 0$, compile to make $\bigoplus_{n \geq 0} H^*(C_n(M))$ into a ring.

The superposition product

- The maps $\mu_{m,n}$, for $m, n \geq 0$, compile to make $\bigoplus_{n\geq 0} H^*(C_n(M))$ into a ring.

Theorem (S. '23)

Both of the following isomorphisms

$$\bigoplus_{n\geq 1} H^*(C_n(M))[2kn] \cong H^*(\Gamma(M, S^{2k}))$$

and

$$\bigoplus_{i,n\geq 0} H^i(C_n(M);\mathbb{Q})[2kn] \cong H^*(\mathrm{map}_\partial(M, S^{d+2k});\mathbb{Q})$$

are, in fact, ring isomorphisms with the cup product on the right and the superposition product on the left.

The superposition product

- The maps $\mu_{m,n}$, for $m, n \geq 0$, compile to make $\bigoplus_{n\geq 0} H^*(C_n(M))$ into a ring.

Theorem (S. '23)

Both of the following isomorphisms

$$\bigoplus_{n\geq 1} H^*(C_n(M))[2kn] \cong H^*(\Gamma(M, S^{2k}))$$

and

$$\bigoplus_{i,n\geq 0} H^i(C_n(M);\mathbb{Q})[2kn] \cong H^*(\mathrm{map}_\partial(M, S^{d+2k});\mathbb{Q})$$

are, in fact, ring isomorphisms with the cup product on the right and the superposition product on the left.

- The proof is completely elementary!

The superposition product

- The maps $\mu_{m,n}$, for $m, n \geq 0$, compile to make $\bigoplus_{n\geq 0} H^*(C_n(M))$ into a ring.

Theorem (S. '23)

Both of the following isomorphisms

$$\bigoplus_{n\geq 1} H^*(C_n(M))[2kn] \cong H^*(\Gamma(M, S^{2k}))$$

and

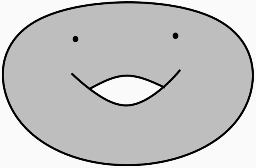
$$\bigoplus_{i,n\geq 0} H^i(C_n(M); \mathbb{Q})[2kn] \cong H^*(\mathrm{map}_\partial(M, S^{d+2k}); \mathbb{Q})$$

are, in fact, ring isomorphisms with the cup product on the right and the superposition product on the left.

- The proof is completely elementary!
- The superposition product is crucial in Moriyama's work.

THANK YOU

and have a good evening!

 $\in C_2(\mathbb{T}^2)$