# Aufgaben zur Topologie

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**Week 3 — Coverings** to be done by: 15.11.2016

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**Exercise 3.1** (Local properties of covering spaces)
A map $f\colon Y \to X$ is a *local homeomorphism* if, for each $y \in Y$, there exists an open neighbourhood $U$ of $y$ such that $f(U)$ is open in $X$ and the restriction $f|_U \colon U \to f(U)$ of $f$ to $U$ is a homeomorphism. Suppose that $f\colon Y \to X$ is a local homeomorphism. Show for each of the following properties that if $X$ has this property, then so does $Y$.
(a) locally connected,
(b) locally path-connected,
(c) locally compact.
Now let $\xi\colon \tilde{X} \to X$ be a covering.
(d) Show that $\xi$ is a local homeomorphism.
Show for each of the following properties that if $X$ has this property, so does $\tilde{X}$.
(e) Hausdorff,
(f) compact, if – in addition – the fibre is finite.

**Exercise 3.2** (A covering space of a manifold is a manifold.)
Let $\xi\colon \tilde{M} \to M$ is a covering with finite or countable fibre. If $M$ is a (differentiable, $C^r$, smooth, holomorphic, ... ) manifold, then so is $\tilde{M}$. If $M$ is orientable, then so is $\tilde{M}$ (however, the reverse implication does not hold).

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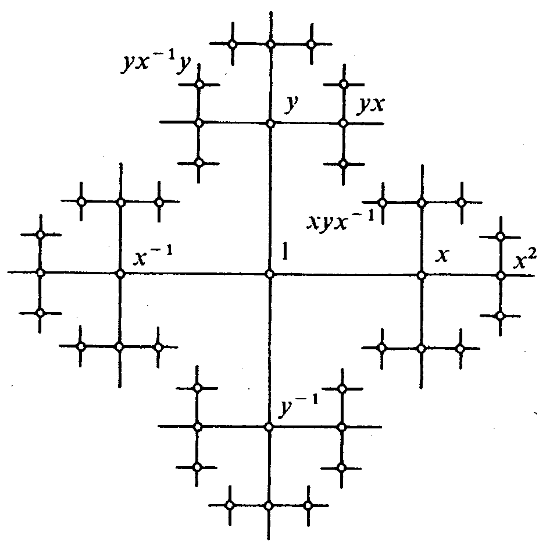


Universal cover of the figure-eight space $\mathbb{S}^1 \vee \mathbb{S}^1$

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**Exercise 3.3** (Properly discontinuous group actions)
Let the discrete group $G$ act on a space $Y$ and denote the action by $(g, y) \mapsto g\,.\,y$. The action is said to be *properly discontinuous*, if for each $y \in Y$ there is a neighbourhood $U$, such that $(g\,.\,U) \cap U = \emptyset$ for all but finitely many $g \in G$. If $Y$ is Hausdorff and the action is free, then this is equivalent to the statement that, for each $y \in Y$, there

is a neighbourhood $U$, such that $(g \centerdot U) \cap U = \emptyset$ for all $g \in G$ except the identity element. (Why is this true?)

(a) For a free and properly discontinuous action on a Hausdorff space $Y$, the quotient map $\xi \colon Y \to X := Y/G$ is a covering with fibre $G$.
(b) Example: $G = \mathbb{Z}$ and $Y = \mathbb{C} - \{0\}$; for the action fix a complex number $\lambda \neq 0$ and set $n \centerdot z := \lambda^n z$. For which $\lambda$ is this action free, for which is it properly discontinuous ? What is the quotient ?
(c) Example: Let $G = \mathrm{Fr}(2) = \langle x, y \,|\; \rangle$ denote the free (non-abelian) group on two letters $x, y$. The figure above shows its Cayley graph $C$, which is a tree and which we regard as a subspace of the plane. The vertices are reduced words $w$ in the letters $x, y$ (and their inverses) and we denote this vertex by $(w)$; an edge between the vertices $(w)$ and $(w')$ we denote by $(w, w')$ and such an edge exists iff $w^{-1} w'$ is $x$ or $y$ or $x^{-1}$ or $y^{-1}$. Thus there is a vertex for each group element, and there are four edges emanating from each vertex. $C$ is contractible and thus simply-connected. The right-action of $G$ on $C$ is described for a $g \in G$ as follows: for a vertex we set $(w) \centerdot g = (wg)$; for an edge we set $(w, w') \centerdot g = (wg, w'g)$. Note that as a continuous map the action by $g$ must dilate the lengths of the edges. Show that the action is free and properly discontinuous. And show that the quotient $C/G$ is the figure-eight space $\mathbb{S}^1 \vee \mathbb{S}^1$. Conclude $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathrm{Fr}(2)$.

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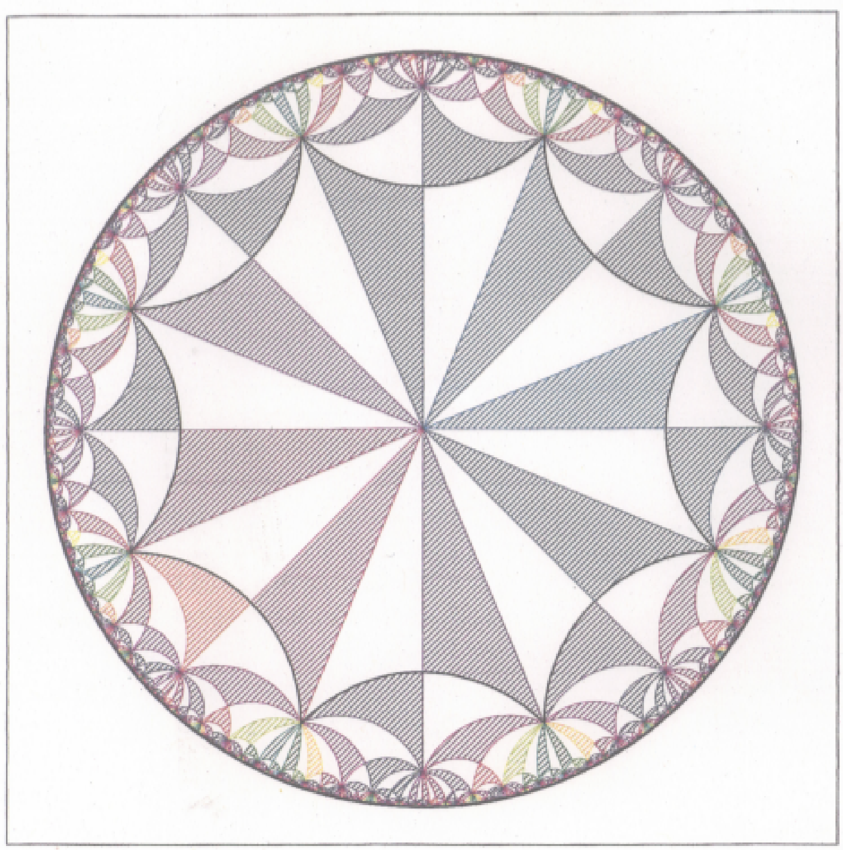
**Figure 12.3.** A tiling of the hyperbolic plane modelled by the circular disk. From the hyperbolic point of view, all the 'triangles' are exactly the same shape. The (orientation-preserving) symmetries are Möbius maps which preserve this tiling and map any coloured triangle onto any other. This and the following figure were drawn by our DFS tiling program with the aid of an automaton provided by the program KBMAG by Derek Holt. The pattern of the colouring relates to the automaton.



A properly discontinuous and free action on the hyperbolic plane $\mathbb{H}^2$. Each semicircle $S$ in the picture (together with a choice of orientation) defines an isometry of $\mathbb{H}^2$ given by translation parallel to $S$ by an amount such that each vertex on $S$ is moved forwards by two steps, i.e., it is sent to the next-but-one vertex in the direction given by the orientation. These hyperbolic translations generate a discrete subgroup $G$ of the group $\mathrm{Isom}(\mathbb{H}^2)$ of all hyperbolic isometries of $\mathbb{H}^2$, which acts properly discontinuously and freely. Image credit: *Indra's Pearls*, David Mumford, Caroline Series and David Wright.

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**Exercise 3.4** (The pull-back of a covering is a covering.)
Let $\xi\colon \tilde{X} \to X$ be a covering and $f\colon Y \to X$ be any map. The *pull-back of $\xi$ along $f$* consists of the space $\tilde{Y} = f^*(\tilde{X}) := \{(y,\tilde{x}) \in Y \times \tilde{X} \mid f(y) = \xi(\tilde{x})\}$ (with the subspace topology of the product topology) together with two maps $\tilde{\xi} = f^*(\xi)\colon \tilde{Y} = f^*(\tilde{X}) \to Y$, defined by $(y,\tilde{x}) \mapsto y$ and $\tilde{f}\colon \tilde{Y} = f^*(\tilde{X}) \to \tilde{X}$, defined by $(y,\tilde{x}) \mapsto \tilde{x}$. Thus we have a commutative square:

$$\begin{tikzcd}
\tilde{Y} = f^*(\tilde{X}) \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\xi}=f^*(\xi)"'] & \tilde{X} \arrow[d, "\xi"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

(a) $\tilde{\xi}\colon \tilde{Y} \to Y$ is a covering with the same fibre as $\xi$.

(b) The following formulae hold, where $g\colon Z \to Y$ and $f\colon Y \to X$:

(1) $g^*(f^*(\tilde{X})) = (f \circ g)^*(\tilde{X})$ and $g^*(f^*(\xi)) = (f \circ g)^*(\xi)$;

(2) $\mathrm{id}^*(\tilde{X}) = \tilde{X}$ and $\mathrm{id}^*(\xi) = \xi$.

(c) If $\xi\colon \mathbb{S}^2 \to \mathbb{R}\mathrm{P}^2$ is the antipodal projection and $f\colon \mathbb{R}\mathrm{P}^2 - \{P\} \to \mathbb{R}\mathrm{P}^2$ the inclusion, then the pull-back $\tilde{\xi}$ is the covering of the Möbius-band by a band.
(d) Let $\xi_n\colon \mathbb{S}^1 \to \mathbb{S}^1$ denote the $n$-fold covering $z \mapsto z^n$. What is $\xi_n^*(\xi_m)$ ? And what is $\xi_n \circ \xi_m$ from Exc. 2.3 ?

**Exercise 3.5** (Maps out of spaces with finite fundamental group)
Let $X$ be a path-connected (locally path-connected) space with basepoint $x_0$ such that $\pi_1(X, x_0)$ is finite.
(a) Show that any continuous map $X \to \mathbb{S}^1$ must be nullhomotopic.
(b) Using part (a), show that any continuous map $X \to \prod^k \mathbb{S}^1$ to a product of $k$ copies of the circle $\mathbb{S}^1$ must be nullhomotopic.
(c) Show that any continuous map $X \to \bigvee^\ell \mathbb{S}^1$ to a wedge of copies of $\ell$ copies of the circle $\mathbb{S}^1$ must be nullhomotopic. (Use Exercise 3.3(c) when $\ell = 2$. For larger values of $\ell$ you will need an appropriate generalisation of that exercise.)

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**Exercise 3.6*** (The fundamental groupoid and fibre-transport)

A *groupoid* is a category in which every morphism is invertible. It earns its name from the following example. If $G$ is a discrete group, then it may be considered as a groupoid $\mathcal{B}G$ with one object $\bullet$ and with the morphism set $\mathrm{Hom}_{\mathcal{B}G}(\bullet,\bullet) = \mathrm{Aut}_{\mathcal{B}G}(\bullet)$ equal to $G$. Another example that one may build from $G$ is $\mathcal{E}G$, which has one object for each element of $G$, and exactly one morphism between any pair of objects (there is then only one possible way in which composition may be defined).

*If we imagine a category $\mathcal{C}$ (and therefore in particular a groupoid) as having a vertex for each object, an edge for each morphism, a triangle for each pair $(f,g)$ of morphisms such that* $\mathrm{source}(f) = \mathrm{target}(g)$*, etc.. One should think of $f$ as a 'connection' between two objects, materialized as an edge, $g$ likewise as another edge and $f \circ g$ as a third edge; and the fact that the third edge stands for the compostion $f \circ g$ and not just for any morphisms, we materilaize as a triangle — etc., etc.. The resulting space is called the classifying space* $\mathrm{B}\mathcal{C}$ *of the category $\mathcal{C}$. For $\mathcal{C} = \mathcal{E}G$ this space turns out to be a simplex of cardinality $|G|$ (which may be infinite).*

A third example, which may be built out of any topological space $X$, is the following. We take one object for each point $x \in X$. A morphism from $x$ to $y$ is then defined to be a homotopy class of paths in $X$ starting at $x$ and ending at $y$, where "homotopy" means homotopy of maps $[0,1] \to X$ relative to $\{0,1\}$. It is obvious, what the identity morphisms are and what the composition of two morphisms is. The result is a groupoid (why ?), denoted $\Pi(X)$.
(1) What is the group of automorphisms of the object $x \in X$ ?

(2) Show that any continuous map $f\colon X \to Y$ induces a functor $\Pi(f)\colon \Pi(X) \to \Pi(Y)$. We have $\Pi(g \circ f) = \Pi(g) \circ \Pi(f)$. Furthermore, if $F\colon X \times [0,1] \to Y$ is a homotopy between the maps $f_0(x) = F(x,0)$ und $f_1(x) = F(x,1)$, the $F$ allows us to define a transformation from the functor $\Pi(f_0)$ to the functor $\Pi(f_1)$.

Now let $\xi\colon \tilde{X} \to X$ be a covering.
(3) Show that the unique path-lifting property allows us to define a "fibre-transport" functor

$$\text{trans}\colon \Pi(X) \longrightarrow \text{Sym},$$

where Sym is the category of sets and bijections, such that a point $x \in X$, i.e., an object of $\Pi(X)$, is taken to its fibre $\xi^{-1}(x)$.