Aufgaben zur Topologie

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Week 11 — Mayer-Vietoris sequence, reduced homology, mapping degree Due: 25. January 2017

Exercise 11.1 (Spaces with finite type homology; homological dimension.)
The homology of a space X is called *of finite type* if there exists an integer $N \geq 0$ such that
(F1) $H_n(X) = 0$ for $n > N$,
(F2) $H_n(X)$ is finitely generated for $0 \leq n \leq N$.

(1) Show, for a space $X = X_1 \cup X_2$, that if X_1 and X_2 are open subspaces of X having homology of finite type, and the homology of $X_1 \cap X_2$ is also of finite type, then so is the homology of X.

The smallest integer N such that (F1) holds is called the *homological dimension* $\operatorname{dimhom}(X)$ of X.
(2) Using the proof of part (1), give an upper bound for $\operatorname{dimhom}(X)$ in terms of $\operatorname{dimhom}(X_1)$, $\operatorname{dimhom}(X_2)$ and $\operatorname{dimhom}(X_1 \cap X_2)$.

Exercise 11.2 (Reduced homology and augmentation.)
Recall that the reduced homology of a space X is defined as $\widetilde{H}_n(X) = \ker\big(e_*\colon H_n(X) \to H_n(P)\big)$, where P is a one-point space and $e\colon X \to P$ the obvious map. Consider the singular chain complex $S_\bullet(X)$ of X and the *augmentation homomorphism*

$$\epsilon\colon S_0(X) \longrightarrow \mathbb{Z} \qquad \epsilon\Big(\sum_i \lambda_i x_i\Big) = \sum_i \lambda_i,$$

where we wrote, instead of a 0-simplex $a_i\colon \Delta^0 \to X$ with $a_i(e_0) = x_i$, just the value x_i. We call

$$S_\bullet^{\mathrm{aug}}(X)\colon \qquad 0 \leftarrow \mathbb{Z} \xleftarrow{\epsilon} S_0(X) \xleftarrow{\partial} S_1(X) \xleftarrow{\partial} \cdots$$

with $S_{-1}^{\mathrm{aug}}(X) = \mathbb{Z}$ the *augmented singular chain complex* of X.

(1) Show that $\epsilon \circ \partial = 0$. So one may regard $S_\bullet^{\mathrm{aug}}(X)$ as a chain complex with one negatively indexed chain group.
(2) There is a chain map $E_\bullet\colon S_\bullet^{\mathrm{aug}}(X) \to S_\bullet(X)$ with $E_n = \mathrm{id}$ for $n \geq 0$ and $E_{-1} = 0$:

$$\begin{tikzcd}
0 & \mathbb{Z} \arrow[l] \arrow[d] & S_0(X) \arrow[l, "\epsilon"'] \arrow[d, equal] & S_1(X) \arrow[l, "\partial"'] \arrow[d, equal] & \cdots \arrow[l] \\
0 & 0 \arrow[l] & S_0(X) \arrow[l] & S_1(X) \arrow[l, "\partial"'] & \cdots \arrow[l]
\end{tikzcd}$$

(3) The chain map $E_\bullet$ induces homomorphisms

$$E_*\colon H_n(S_\bullet^{\mathrm{aug}}(X)) \longrightarrow H_n(S_\bullet(X))$$

for each n. Show that E_* is injective for $n \geq 0$, an isomorphism for $n \geq 1$ and $H_{-1}(S_\bullet^{\mathrm{aug}}(X)) = 0$ for non-empty X.
Hint: Regard ϵ as a chain map

$$\begin{tikzcd}
0 & S_0(X) \arrow[l] \arrow[d, "\epsilon"'] & S_1(X) \arrow[l, "\partial"'] \arrow[d] & S_2(X) \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l] \\
0 & \mathbb{Z} \arrow[l] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

(4)* Let $A_\bullet$ be any chain complex and "split" it into a negative part ($A_n^- = A_{n-1}$ for $n \leq 0$ and $A_n^- = 0$ for $n > 0$) and a non-negative part ($A_n^+ = 0$ for $n < 0$ and $A_n^+ = A_n$ for $n \geq 0$). We have a chain map $D_\bullet\colon A_\bullet^+ \to A_\bullet^-$ as follows (where $A_\bullet$ is "bent"):

$$\begin{tikzcd}
\cdots & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & A_0 \arrow[l] \arrow[d, "\partial"'] & A_1 \arrow[l, "\partial"'] \arrow[d] & A_2 \arrow[l, "\partial"'] \arrow[d] & \cdots \arrow[l, "\partial"'] \\
\cdots & A_{-3} \arrow[l, "\partial"] & A_{-2} \arrow[l, "\partial"] & A_{-1} \arrow[l, "\partial"] & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

with $D_n = 0$ for $n \neq 0$ and $D_0 = \partial\colon A_0 \to A_{-1}$. Thus we have induced homomorphisms

$$D_*\colon H_n(A_\bullet^+) \longrightarrow H_n(A_\bullet^-).$$

(a) $H_n(A_\bullet^+) = 0$ for $n < 0$
(b) $H_n(A_\bullet^+) = H_n(A_\bullet)$ for $n \geq 1$
(c) $H_n(A_\bullet^-) = 0$ for $n > 0$
(d) $H_n(A_\bullet^-) = H_{n-1}(A_\bullet)$ for $n \leq -1$
(e) $H_0(A_\bullet) \cong \ker\big(D_*\colon H_0(A_\bullet^+) \to H_0(A_\bullet^-)\big)$
(f) $H_{-1}(A_\bullet) \cong \operatorname{coker}\big(D_*\colon H_0(A_\bullet^+) \to H_0(A_\bullet^-)\big)$
What if we "bend" $A_\bullet$ several times?

Exercise 11.3 (Moore spaces.)
(a) Let $f\colon X \to Y$ be a continuous map. Recall from Exercise 8.3 the definition of the *cone* of f, denoted $\mathrm{Cone}(f)$, and that there is an embedding $Y \hookrightarrow \mathrm{Cone}(f)$ of Y as a closed subspace of $\mathrm{Cone}(f)$.
(b) Show that the image of Y under this embedding has an open neighbourhood that deformation retracts onto it. Therefore use the long exact sequence for the pair $(\mathrm{Cone}(f), Y)$ and Exercise 10.4 (the Quotient Theorem) to construct a long exact sequence of the form:

$$\cdots \to H_n(Y) \to H_n(\mathrm{Cone}(f)) \to \widetilde{H}_n(\Sigma X) \to H_{n-1}(Y) \to H_{n-1}(\mathrm{Cone}(f)) \to \cdots.$$

(c) Use the Suspension Isomorphism, and analyse the above sequence carefully in degree zero, to show that we have a long exact sequence of the form

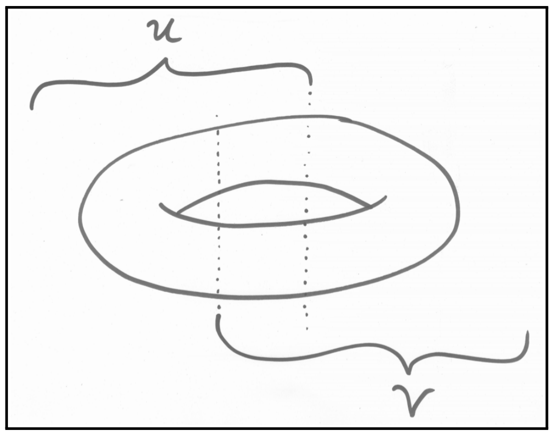
$$\cdots \to \widetilde{H}_n(Y) \to \widetilde{H}_n(\mathrm{Cone}(f)) \to \widetilde{H}_{n-1}(X) \to \widetilde{H}_{n-1}(Y) \to \widetilde{H}_{n-1}(\mathrm{Cone}(f)) \to \cdots,$$

where the map $\widetilde{H}_{n-1}(X) \to \widetilde{H}_{n-1}(Y)$ in the sequence is the one induced by f.
(d) Apply this sequence to a map $f_{n,k}\colon \mathbb{S}^n \to \mathbb{S}^n$ of degree k, where $n > 0$ and $k \neq 0$, to show that $H_i(\mathrm{Cone}(f_{n,k}))$ is isomorphic to $\mathbb{Z}/k\mathbb{Z}$ for $i = n$ and is zero for $i \neq n$.
(e) Now let $n > 0$ and let A be any finitely generated abelian group. Construct a space Y such that $\widetilde{H}_n(Y) \cong A$ and $\widetilde{H}_i(Y) = 0$ for $i \neq n$. Such a space is called a *Moore space* for the pair (A, n).
(*Hint*: look at Exercise 11.5 part (1) below.)
In fact a similar construction works for any (not necessarily finitely generated) abelian group A.
(f) Now let $(A_0, A_1, A_2, \ldots)$ be any sequence of finitely generated abelian groups with A_0 free. Construct a space Y such that $H_i(Y) \cong A_i$ for all $i \geq 0$.

Exercise 11.4 (The degree of a rotation.)
Let $A \in O(n+1)$. The restriction $f = A|_{\mathbb{S}^n}$ of $A\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a self-map of $\mathbb{S}^n$, and therefore has a degree, defined as in the lectures to be the unique integer $\deg(f)$ such that $f_*(x) = \deg(f).x$ where x is any non-zero element of $H_n(\mathbb{S}^n) \cong \mathbb{Z}$. (We assume that $n > 0$ in this exercise.)
(1) Explain why $f = A|_{\mathbb{S}^n}$ must have degree either $+1$ or -1.
(2) Show that, in fact, $\deg(A|_{\mathbb{S}^n}) = \det(A)$.
(*Hint*: first construct an explicit singular n-cycle on $\mathbb{S}^n$ representing a non-zero element of $H_n(\mathbb{S}^n)$.)
(3) In particular, if A is a reflection in a k-dimensional subspace in $\mathbb{R}^{n+1}$, $\deg(A|_{\mathbb{S}^n}) = (-1)^{n-k+1}$.

Exercise 11.5 (Applications of Mayer-Vietoris.)
(1) Show that for a wedge of spaces $X \vee Y$ we have $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for $n > 0$. What happens in degree zero? Thus compute the homology of $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$.
(2) Decompose $\mathbb{S}^1 \times \mathbb{S}^1$ into two open subsets, each homeomorphic to the open annulus $\mathbb{S}^1 \times (0, 1)$, such that their

intersection is the disjoint union of two open annuli (see the figure on the next page). Using the Mayer-Vietoris sequence for this decomposition, compute the homology of $\mathbb{S}^1 \times \mathbb{S}^1$. (Cf. your answer to Exercise 10.5(c).)
(3) Note that these two spaces have isomorphic homology in each degree. Explain (using π_1) why, nevertheless, they are *not* homotopy equivalent spaces.
(4) Describe the universal coverings of $\mathbb{S}^1 \times \mathbb{S}^1$ and of $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ and compute the homology groups of each.
(5)* Let Y be the torus $\mathbb{S}^1 \times \mathbb{S}^1$ and let Z be the Möbius band, with $A \subset Z$ its boundary. Let $\phi\colon A \cong \mathbb{S}^1$ be a parametrisation of A. Define an embedding $g\colon A \hookrightarrow Y$ by $g(a) = (\phi(a), 1)$, where $1 \in \mathbb{S}^1$ is the basepoint. Let $X = Z \cup_g Y$, as defined in Exercise 8.3. Using an open cover by "small" open neighbourhoods of $Y \subset X$ and $Z \subset X$, compute the homology of X in all degrees.
(6)* Repeat the previous exercise, but instead of the embedding g, use the embedding $h_k\colon A \hookrightarrow Y$ defined by $h_k(a) = (\phi(a), \phi(a)^k)$, where k is a fixed integer. (Note that $h_0 = g$.) Compute the homology of $X_k = Z \cup_{h_k} Y$.



The decomposition of the torus in Exercise 11.5(2).

Exercise 11.6* (The Brouwer Fixed Point Theorem and PageRank.)
(a) Let n be a positive integer and let $f\colon \mathbb{D}^n \to \mathbb{D}^n$ be a self-map of the n-dimensional disc. Show that f has a fixed point.
(*Hint*: the one-dimensional case can be proved using the Intermediate Value Theorem. You have already seen a proof of the two-dimensional case, using π_1. The proof for $n > 2$ is very similar, using homology instead.)
(b) So any self-map of a space homeomorphic to $\mathbb{D}^n$ has (at least) one fixed point.
(c) (Theorem of Perron-Frobenius) Let A be a real $n \times n$ matrix. If all of its entries are non-negative, then it has a non-negative eigenvalue and a corresponding eigenvector all of whose entries are non-negative. Prove this statement in two cases: if there is a non-zero vector x with non-negative entries such that $Ax = 0$ then we are done; otherwise consider the self-map f of $\Delta^{n-1} \subseteq \mathbb{R}^n$ defined by

$$f(x) = \tfrac{y}{y_1 + \cdots + y_n} \qquad \text{for} \qquad y = (y_1, \ldots, y_n) = Ax.$$

(First explain why this is well-defined.)
(d) (Stronger version for column-sum-one matrices.) Suppose that A has the property that the sum of the entries in each column is 1. Using similar methods to part (c), show that, in this case, we can take the eigenvector to lie in Δ^{n-1} and the eigenvalue to be equal to 1.
(e) (Solution to the PageRank problem.) Let G be a directed graph with a vertex for every webpage and an edge from w_1 to w_2 if the webpage w_1 contains a link to the webpage w_2. We want to extract from this a measure of the relative "importance" of each webpage. This will assign to each webpage w a non-negative real number $a(w)$, such that the sum of $a(w)$ over all webpages w is 1. In other words, a is a point on the $(n-1)$-simplex $\Delta^{n-1} \subseteq \mathbb{R}^n$, where n is the number of webpages on the internet.

The first idea is to simply set $a(w)$ equal to the number of other webpages that link to w, and then normalise so that the sum is equal to 1. But then webpages that link to millions of other webpages would have a disproportionate influence on the results compared to those with relatively few links. So to make the measure more democratic, we say that each webpage has one "vote", which it distributes to other webpages by linking to them: if it links to 5 other webpages, each of those webpages earns $\frac{1}{5}$ of its vote. Then $a(w)$ is equal to the total number of votes received by w from other websites – again we have to normalise this. The final version of the idea is to give a higher weight to the votes from websites that are themselves more "important" according to the measure a. This sounds circular, but what it means is just that, instead of a direct definition of the vector $a \in \Delta^{n-1} \subseteq \mathbb{R}^n$, we have a system of linear equations that it must satisfy. Namely:

$$a(w_i) \ = \ \sum_{j=1}^{n} A(i,j).a(w_j),$$

where $A(i,j) = 0$ if there is no edge $w_j \to w_i$ and $A(i,j) = \frac{1}{k}$ if there is an edge $w_j \to w_i$ and k is the total number of edges leaving the vertex w_j (i.e., the total number of distinct links on the webpage w_j). If we let A be the matrix whose (i,j)th entry is $A(i,j)$, then these equations are equivalent to:

$$a = Aa.$$

The PageRank problem is to find such a vector $a \in \Delta^{n-1} \subseteq \mathbb{R}^n$. This gives us our measure of the relative "importance" of each webpage, and (ignoring the issue that some entries of the vector a might be equal!) we may use this to rank all webpages by importance.
To do: show that the PageRank problem has a solution.
(f) (Uniqueness of the solution in a strongly connected internet.) Let us make the unreasonable assumption that the internet is strongly connected: for any pair of webpages (w_1, w_2) there exists a sequence of links taking you from w_1 to w_2. Prove that the vector $a \in \Delta^{n-1}$ such that $a = Aa$ is unique in this case.
(*Hint*: Do this by contradiction. Non-uniqueness of this solution means that the 1-eigenspace of A intersects Δ^{n-1} in more than one point, so it must have dimension at least 2. But then it must also intersect the boundary of Δ^{n-1}, so there exists a solution a such that $a(w_i) = 0$ for some website w_i. Show that this implies that $a(w_j) = 0$ for every website w_j that links to w_i.)
(g)** What happens for a less well-connected internet?

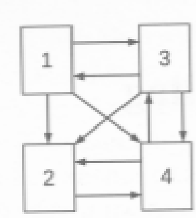


Abb. 1.1 Verknüpfungsstruktur in einem „4-Seiten-Internet"

From the introduction to the book *Lineare Algebra* by J. Liesen and V. Mehrmann. Note that this 4-page internet is strongly connected (cf. part (f) above), so there is a unique solution to the PageRank problem in this case. Can you compute it?