Exotic spheres

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Abstract

These are partial notes from the course on *Exotic Spheres* that I am teaching this semester (Sommersemester 2017) in Bonn. The course webpage is: zatibq.com/exotic-spheres.
So far I have written these up in detail up to the end of lecture 1, together with sketches of what we covered in lectures 2–6 (the sketches are more detailed for lectures 5 and 6). Detailed references still need to be added.
Let me know if you spot any mistakes!

Contents

1. Lecture 1 – Introduction and overview.

Assumption 1.1 Unless stated otherwise, all manifolds in this course will be assumed to be smooth (equipped with a choice of C^∞ atlas), connected, compact, oriented and without boundary. The most frequent exception will be cobordisms, which of course are allowed to have non-empty boundary. Compact and without boundary will usually be abbreviated to *closed*.

Note: we assume that the manifolds are orient*ed*, not just orient*able*, i.e., we assume that $H_n(M;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, where M is the manifold and n is its dimension, and that we have chosen a generator, which we denote $[M]$ and call the *fundamental class*. Second note: for manifolds, *connected* is equivalent to *path-connected*, since they are always locally path-connected (being locally homeomorphic to Euclidean space).

Diffeomorphisms $f\colon M \to N$ between manifolds will by default be assumed to be orientation-preserving, in other words (or rather in symbols): $f_*([M]) = [N] \in H_n(N;\mathbb{Z})$.

Note It is not really significant that we are taking *smooth* to mean C^∞ rather than, say C^1 or C^{12} or even C^ω (analytic). For $r \geqslant 1$, every paracompact C^r manifold M admits a unique compatible

C^s structure for each $r \leqslant s \leqslant \infty$. Here, *compatible* means that the transition functions between any chart in the maximal C^s atlas and any chart in the maximal C^r atlas are differentiable of type C^r, and unique means that any two compatible choices of C^s structure on M are C^s-diffeomorphic. This is a result of Whitney. Moreover, a theorem of Morrey-Grauert says that any paracompact C^∞ manifold admits a unique compatible C^ω structure.

Thus, schematically, we have:

$$C^0 \qquad C^1 \rightsquigarrow C^2 \rightsquigarrow C^3 \rightsquigarrow \qquad \cdots \qquad \rightsquigarrow C^\infty \rightsquigarrow C^\omega \tag{1.1}$$

where the notation $X \rightsquigarrow Y$ means that an X uniquely determines a Y up to Y-isomorphism. This lecture course is, effectively, about the gap between C^0 and C^1 in the case of spheres, but it is usually is more convenient to assume that things are infinitely differentiable, so we will actually talk about the gap between C^0 and C^∞.

Definition 1.2 To fix notation, we define:

$$\begin{aligned} \mathbb{S}^{n-1} &= \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\} \\ \mathbb{D}^n &= \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leqslant 1\}, \end{aligned}$$

which are manifolds (with boundary, in the case of $\mathbb{D}^n$) and $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$.

Definition 1.3 A *topological sphere* is a manifold M that is homeomorphic to $\mathbb{S}^n$ for $n = \dim(M)$. A topological sphere that is *not diffeomorphic* to $\mathbb{S}^n$ is called an *exotic sphere*. A *homotopy sphere* is a manifold that is homotopy-equivalent to $\mathbb{S}^n$ for $n = \dim(M)$.

Remark 1.4 Using the Hurewicz theorem and Whitehead's theorem, we see that an equivalent condition for a manifold M to be a homotopy sphere (in dimensions $\geqslant 2$) is $\pi_1(M) = 0$ and $\widetilde{H}_i(M) = 0$ for $i < n$.

Definition 1.5 The *connected sum* of two manifolds M_1 and M_2, denoted $M_1 \sharp M_2$, is the manifold obtained by cutting out the interior of an embedded disc $\mathbb{D}^n$ from each of the manifolds, and gluing the two boundaries by an orientation-preserving diffeomorphism. We will define this more carefully in the next lecture, in a way that makes it clear that we again obtain a canonical smooth structure on $M_1 \sharp M_2$, and we will show that this construction, up to orientation-preserving diffeomorphism, is independent of the choices of embedded discs.

A related operation is the *boundary connected sum* of two manifolds with non-empty boundary. This is denoted $M_1 \natural M_2$, and is obtained by choosing an embedded disc $\mathbb{D}^{n-1}$ in each of ∂M_1 and ∂M_2 and gluing them together via an orientation-preserving diffeomorphism. Again, this can be defined more carefully, giving $M_1 \natural M_2$ a well-defined smooth structure, and the result is independent, up to orientation-preserving diffeomorphism, of the choices of embedded discs – as long as ∂M_1 and ∂M_2 are connected. A subtlety with this construction is that, if M_1 or M_2 has several boundary components, then the boundary connected sum $M_1 \natural M_2$ depends (a priori) on choosing a boundary component for each manifold.

1.1. Monoids of n-manifolds

Definition 1.6 Let $\mathcal{M}_n$ be the set[1] of diffeomorphism classes of n-manifolds (with the properties of Assumption 1.1), equipped with the operation $\sharp\colon \mathcal{M}_n \times \mathcal{M}_n \longrightarrow \mathcal{M}_n$. It is easy to check that $\sharp$ is associative and commutative, and that $M \sharp \mathbb{S}^n \cong M$ for any M. Hence $(\mathcal{M}_n, \sharp)$ is a commutative monoid with unit element $\mathbb{S}^n$.

Lemma 1.7 *If Σ and Σ' are homotopy spheres, then so is $\Sigma \sharp \Sigma'$.*

[1] By Whitney's embedding theorem, every smooth, paracompact manifold may be embedded into $\mathbb{R}^\infty$, so every manifold is diffeomorphic to a subset of $\mathbb{R}^\infty$ and so the collection of diffeomorphism classes forms a set, not a proper class – in fact it has cardinality at most $2^{2^{\aleph_0}}$. Actually, this is not technically correct, and one can only say that $\mathcal{M}_n$ is *in bijection with* a set, since its elements (diffeomorphism classes) are not sets, but proper classes. But I'm not a set theorist, so I won't worry about this too much, and I will probably often say "set" when what I am talking about is really a proper class.

We will prove this tomorrow. This lemma tells us that the subset $\mathcal{S}_n \subset \mathcal{M}_n$ consisting of homotopy n-spheres is closed under the operation $\sharp$, and is therefore a submonoid. An even easier fact is that the connected sum of two topological n-spheres is again a topological n-sphere, so the subset $\mathcal{T}_n \subset \mathcal{M}_n$ consisting of topological n-spheres is also a submonoid.

Definition 1.8 Given two n-manifolds M and N, a *cobordism* from M to N is an $(n+1)$-manifold W with boundary, together with a decomposition of its boundary $\partial W = \partial_0 W \sqcup \partial_1 W$ as a disjoint union, an orientation-preserving diffeomorphism $M \to \partial_0 W$ and an orientation-reversing diffeomorphism $N \to \partial_1 W$. It is an *h-cobordism* if both composites $M \to \partial_0 W \subset W$ and $N \to \partial_1 W \subset W$ are homotopy equivalences.

There is an equivalence relation $\sim$ on the set of n-manifolds in which two manifolds M, N are equivalent if and only if there exists an h-cobordism between them – they are then called *h-cobordant*. Reflexivity follows from the fact that $M \times [0,1]$ is an h-cobordism from M to itself; if W is an h-cobordism from M to N, then $\overline{W}$ (the manifold W with its orientation reversed) is an h-cobordism from N to M, so the relation is symmetric; if W, W' are h-cobordisms from L to M and M to N respectively, then by gluing W to W' along $\partial_1 W \to M \to \partial_0 W'$ we obtain an h-cobordism from L to N, so the relation is transitive.

We note that if M and N are diffeomorphic, they are also h-cobordant. Suppose that $\phi\colon M \to N$ is a diffeomorphim. Take $W = N \times [0,1]$, together with the decomposition of its boundary $\partial_0 W = N \times \{0\}$ and $\partial_1 W = N \times \{1\}$ and the diffeomorphisms $x \mapsto (\phi(x), 0)\colon M \to \partial_0 W$ (orientation-preserving) and $y \mapsto (y, 1)\colon N \to \partial_1 W$ (orientation-reversing).[2] Then W is an h-cobordism from M to N.

This means that $\sim$ gives a well-defined equivalence relation on $\mathcal{M}_n$, the set of diffeomorphism-classes of n-manifolds.

If M is h-cobordant to a homotopy n-sphere Σ via W, then $M \simeq W \simeq \Sigma \simeq \mathbb{S}^n$, so M is also a homotopy n-sphere. Thus the equivalence relation on $\mathcal{M}_n$ restricts to the submonoid $\mathcal{S}_n \subset \mathcal{M}_n$. Let Θ_n denote the set of equivalence classes – called *h-cobordism classes* – of the equivalence relation $\sim$ on $\mathcal{S}_n$.

Lemma 1.9 *If M, M', N, N' are simply-connected manifolds and we have h-cobordisms $M \sim M'$ and $N \sim N'$, then*

$$M \sharp N \ \sim\ M' \sharp N'.$$

We will prove this in a couple of lectures' time. It tells us that:

Corollary 1.10 *There is a well-defined commutative monoid structure on Θ_n given by*

$$[\Sigma] \cdot [\Sigma'] = [\Sigma \sharp \Sigma'].$$

By abuse of notation, we will denote this operation $\cdot$ also by $\sharp$.

Proof. In dimension 1, there is only one homotopy sphere, namely $\mathbb{S}^1$ itself, so Θ_1 is a one-element set, and there is nothing to check. In dimensions at least 2, homotopy spheres are simply-connected, so Lemma 1.9 tells us that varying the choices of representatives of $[\Sigma]$ and $[\Sigma']$ (by an h-cobordism) only changes the connected sum by an h-cobordism, so the operation is well-defined. Associativity, commutativity and the fact that $[M] \sharp [\mathbb{S}^n] = [M]$ follow from the fact that $\mathcal{S}_n$ is a commutative monoid. □

The quotient map $q_n\colon \mathcal{S}_n \longrightarrow \Theta_n$ taking a homotopy n-sphere to its h-cobordism class is a homomorphism. By the discussion so far, we have the following diagram of commutative monoids.

[2] See footnote 4 on page 6.

$$\begin{array}{ccc} \mathcal{M}_n & & \\ \cup\!\!\shortmid & & \\ \mathcal{S}_n & \xrightarrow{q_n} & \Theta_n \\ \cup\!\!\shortmid & & \\ \mathcal{T}_n & & \end{array} \tag{1.2}$$

An important fact, which we will prove soon, is:

Lemma 1.11 *The monoid Θ_n is a group.*

The main theorem of Kervaire and Milnor [KM63] is that Θ_n is in fact a *finite* group.

1.2. Very low dimensions

In dimensions 1 and 2, every topological (i.e., C^0) manifold has a unique smooth structure (this was proved by Radó in 1925). Under the hypotheses of Assumption 1.1, there is only one 1-manifold up to orientation-preserving diffeomorphism, namely $\mathbb{S}^1$, so $\mathcal{M}_1$ is the trivial monoid $\{\mathbb{S}^1\}$. (Note: $\mathbb{S}^1$ with the opposite orientation is not a *different* 1-manifold, because $\mathbb{S}^1$ admits an orientation-reversing diffeomorphism, so it is isomorphic to its opposite.) The classification of surfaces is a classical result, which tells us that $\mathcal{M}_2$ is the free monoid on one generator (the torus), in other words it is isomorphic to $\mathbb{N}$. If we drop the orientability condition, then we instead get the commutative monoid with the presentation $\langle T, P \mid TP = P^3 \rangle$. The only homotopy 2-sphere is $\mathbb{S}^2$, so $\mathcal{S}_2$ (and therefore also Θ_2) is the trivial monoid (group).

1.3. The Poincaré conjecture

The topological Poincaré conjecture in dimension n is the assertion that

tPC. If M is a closed, topological manifold such that $M \simeq \mathbb{S}^n$, then M is homeomorphic to $\mathbb{S}^n$.

The corresponding statement for smooth manifolds is the *smooth Poincaré conjecture*:

sPC. If M is a closed, smooth manifold such that $M \simeq \mathbb{S}^n$, then M is diffeomorphic to $\mathbb{S}^n$.

The weak topological Poincaré conjecture is the same as **tPC**, except that we assume that M admits a smooth structure:

wtPC. If M is a closed, smoothable manifold such that $M \simeq \mathbb{S}^n$, then M is homeomorphic to $\mathbb{S}^n$.

("Smoothable" means that M admits at least one smooth structure, but it is not relevant which one we choose, since the statement only involves homotopy equivalences and homeomorphisms.)

There are obvious implications

$$\textbf{tPC} \Rightarrow \textbf{wtPC} \Leftarrow \textbf{sPC}.$$

Note also that **sPC** in dimension n is equivalent to the assertion that $\mathcal{S}_n$ is the trivial monoid, and **wtPC** in dimension n is equivalent to the assertion that $\mathcal{T}_n = \mathcal{S}_n$.

The cardinality $|\mathcal{T}_n|$ of the monoid $\mathcal{T}_n$ is the number of (non-diffeomorphic) oriented smooth structures on $\mathbb{S}^n$. I said *oriented* smooth structures, since $|\mathcal{T}_n|$ counts the number of *oriented* topological n-spheres up to *orientation-preserving* diffeomorphism. If Σ is an unoriented topological n-sphere, it will be counted twice (once for each orientation) ... unless it admits an orientation-reversing self-diffeomorphism – like for example the standard sphere $\mathbb{S}^n$, for which one can use a reflection in a hyperplane – in which case it is counted just once. We will see in a moment that if $n \neq 4$, the monoid $\mathcal{T}_n$ is in fact a group (and isomorphic to Θ_n), with the inverse of a manifold M given by $\overline{M}$, i.e., M with the opposite orientation. Thus:

$$\text{Number of (unoriented) smooth structures on } \mathbb{S}^n \;=\; \tfrac{1}{2}(|\mathcal{T}_n| + o_2 + 1), \tag{1.3}$$

where o_2 is the number of elements of $\mathcal{T}_n$ of order exactly 2. Hence:

$$\begin{aligned}\text{Number of unoriented exotic } n\text{-spheres} &= \tfrac{1}{2}(|\mathcal{T}_n| + o_2 - 1) \\ \text{Number of oriented exotic } n\text{-spheres} &= |\mathcal{T}_n| - 1.\end{aligned}$$

1.4. Dimensions three and four

In dimension 3, it is still true that every topological manifold has a unique smooth structure (this was proved by Moise in 1952, and also by Cerf in 1968). As a consequence, the topological and smooth Poincaré conjectures are equivalent: $\mathbf{tPC}_3 \equiv \mathbf{sPC}_3$, and the monoid $\mathcal{T}_3$ is trivial. As you have probably heard, G. Perelman (following and completing the programme of R. S. Hamilton) proved the 3-dimensional Poincaré conjecture in 2003, so $\mathcal{S}_3$ is also the trivial monoid.

In dimension 4, very little is known. It is not true that every topological 4-manifold has a unique smooth structure: there are some with none (e.g. the so-called "E_8 manifold") and some with many (e.g. $\mathbb{R}^4$ has uncountably many pairwise non-diffeomorphic smooth structures). So the topological and the smooth Poincaré conjectures are very different. In fact, the topological 4-dimensional Poincaré conjecture is true – it was proved by Freedman in 1982 – but the smooth 4-dimensional Poincaré conjecture is still an open question. Thus we have $\mathcal{T}_4 = \mathcal{S}_4$ but we have no idea what this monoid looks like – no-one knows if it is trivial, or even whether it is a group. We will see later in the course that Θ_4 is the trivial group. However, unlike in every other dimension, the quotient homomorphism $q_4 \colon \mathcal{S}_4 \longrightarrow \Theta_4$ is not known to be an isomorphism, so this does not tell us anything about $\mathcal{S}_4$.

1.5. High dimensions

The word "high" usually means "$\geqslant 5$" when talking about manifolds. The reason for this is that many constructions start working only once the dimension is at least 5. The first and most important example of this is Smale's h-cobordism theorem. The setup for this is as follows. Let M and N be n-manifolds and W be an h-cobordism from M to N. In other words, it is an $(n+1)$-manifold with boundary $\partial W = \partial_0 W \sqcup \partial_1 W$ and chosen diffeomorphisms $M \to \partial_0 W$ and $N \to \partial_1 W$ that preserve and reverse orientation respectively, and such that $M \to \partial_0 W \subset W$ and $N \to \partial_1 W \subset W$ are homotopy equivalences.

Theorem 1.12 (Smale, 1962) *Suppose that $n \geqslant 5$ and that M is simply-connected (hence W and N are also simply-connected). Then there exists a diffeomorphism*

$$\phi \colon W \longrightarrow \partial_0 W \times [0,1] \tag{1.4}$$

whose restriction to $\partial_0 W \subset W$ is the obvious inclusion $\partial_0 W \hookrightarrow \partial_0 W \times [0,1]$ given by $x \mapsto (x,0)$.

Remark 1.13 The assumption that M is simply-connected can be weakened. The *s-cobordism theorem* (proved soon afterwards independently by Barden, Mazur and Stallings) implies that, instead of assuming that M is simply-connected, it is enough to assume that the embedding $M \to \partial_0 W \subset W$ is not just a homotopy equivalence, but a *simple homotopy equivalence*, which is a stronger condition. A homotopy equivalence between simply-connected spaces is always a simple homotopy equivalence, so the h-cobordism theorem is a special case of the s-cobordism theorem.

Corollary 1.14 *The weak topological Poincaré conjecture is true in dimensions at least 5. In other words, $\mathcal{T}_n = \mathcal{S}_n$ for $n \geqslant 5$.*

We will prove this (assuming Theorem 1.12) in a couple of lectures' time. In dimensions at least 6, it follows directly, whereas for dimension 5 it needs an extra input, which is something that we will prove along the way as we go through Kervaire and Milnor's proof that Θ_n is finite for $n \geqslant 4$.

So, in high dimensions, the weak topological Poincaré conjecture (where we assume to begin with the the manifold M is smoothable) is true. This of course does not automatically imply the full topological Poincaré conjecture, where we do not assume that M is smoothable, because, a priori, there might exist topological homotopy n-spheres that do not admit any smooth structure. Actually, there do not exist such things, but to see that they do not exist requires the topological Poincaré conjecture,[3] so it would be circular to try to use this fact to deduce the full topological Poincaré conjecture! But later work of Stallings, Zeeman, Newman and Connell extended Smale's work to prove the full topological Poincaré conjecture, so:

$$\mathbf{tPC}_n \text{ is true for } n \geqslant 5.$$

As we saw above, it is also true in dimensions $n = 1, 2$ (by the classification of 1- and 2-manifolds) and $n = 3$ (by Perelman) and $n = 4$ (by Freedman). So, in fact:

$$\mathbf{tPC}_n \text{ is true for all } n.$$

Another corollary (this one is immediate) of Theorem 1.12 is that $q_n : \mathcal{S}_n \longrightarrow \Theta_n$ is an isomorphism for $n \geqslant 5$. To see this, let Σ and Σ' be two homotopy n-spheres that are h-cobordant via an h-cobordism W. Then we have a diffeomorphism between them, given by

$$\Sigma \longrightarrow \partial_0 W \;\cong\; \partial_0 W \times \{1\} \xrightarrow{(\phi^{-1})|_{\partial_0 W \times \{1\}}} \partial_1 W \longrightarrow \Sigma', \tag{1.5}$$

where ϕ is the orientation-preserving diffeomorphism of Theorem 1.12. The first and third diffeomorphisms are orientation-preserving, whereas the second and fourth are orientation-reversing,[4] so overall this is an orientation-preserving diffeomorphism between Σ and Σ', so they represent the same element of $\mathcal{S}_n$. Thus q_n is injective. It is surjective by construction, so it is an isomorphism of monoids. Thus

$$\mathcal{T}_n = \mathcal{S}_n \cong \Theta_n$$

for $n \geqslant 5$. In particular, $\mathcal{T}_n = \mathcal{S}_n$ is an abelian group (by Lemma 1.11).

As a reminder, for $n \leqslant 3$ we have $\mathcal{T}_n = \mathcal{S}_n = \Theta_n = \{1\}$ and for $n = 4$ we have $\Theta_4 = \{1\}$ and $\mathcal{T}_4 = \mathcal{S}_4$, but nothing is known about the monoid $\mathcal{T}_4$. If the h-cobordism theorem (Theorem 1.12) were true also for $n = 4$, then it would imply that $q_4 : \mathcal{T}_4 = \mathcal{S}_4 \longrightarrow \Theta_4 = \{1\}$ is also an isomorphism, and therefore that $\mathcal{S}_4$ is trivial (the smooth 4-dimensional Poincaré conjecture). However, Donaldson (in 1983) has shown that the h-cobordism theorem is *false* for $n = 4$, so this strategy cannot work.

The main theorem of this lecture course will be:

Theorem 1.15 ([KM63]) *The abelian group Θ_n is finite for $n \geqslant 4$.*

Moreover, Kervaire and Milnor's method of proof gives a way to (in principle) calculate the group Θ_n explicitly in terms of the stable homotopy groups of spheres (and a little extra information, such as in which dimensions there exist manifolds with non-trivial Kervaire invariant). For $n \geqslant 5$, the smooth Poincaré conjecture $\mathbf{sPC}_n$ is true if and only if $\Theta_n = \{1\}$. The current state-of-the-art appears to be that the group Θ_n is known explicitly at least for $1 \leqslant n \leqslant 63$, and in that range the only times that Θ_n is trivial is in dimensions $n = 1, \ldots, 6, 12, 56, 61$. It is also known that Θ_n is never zero in *odd* dimensions $n \geqslant 63$.

As an immediate corollary (using our earlier discussion), it follows that the number of smooth structures on $\mathbb{S}^n$ is finite for $n \geqslant 5$. This result was later extended by Kirby and Siebenmann to any closed, topological manifold M of dimension $\geqslant 5$: the number of (pairwise non-diffeomorphic) smooth structures on M is finite.

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[3] The argument is as follows. Let Σ be a topological homotopy n-sphere, i.e., a topological, closed, oriented, connected n-manifold with the homotopy type of $\mathbb{S}^n$, with $n \geqslant 5$. Using $\mathbf{tPC}_n$ we know that there is a homeomorphism $\theta : \Sigma \to \mathbb{S}^n$. Pulling back the standard smooth structure on $\mathbb{S}^n$ along θ gives us a smooth structure on Σ. Hence Σ is smoothable.

[4] In general, for a manifold M, the obvious diffeomorphism $x \mapsto (x, 0) : M \to M \times \{0\}$, where $M \times \{0\}$ is oriented as part of the boundary of $M \times [0, 1]$, is orientation-*preserving*. On the other hand, the obvious diffeomorphism $x \mapsto (x, 1) : M \to M \times \{1\}$, where $M \times \{1\}$ is also oriented as part of the boundary of $M \times [0, 1]$, is orientation-*reversing*. This is essentially because the outward-pointing normal vector field on $M \times \{1\} \subset M \times [0, 1]$ "points in the opposite direction to" the outward-pointing normal vector field on $M \times \{0\} \subset M \times [0, 1]$.

1.6. Exotic Euclidean spaces

By contrast with spheres (where we will see that there are many exotic smooth structures in higher dimensions – although only finitely many in each dimension), the Euclidean spaces $\mathbb{R}^n$ behave much more simply for $n \neq 4$. Firstly, when $n \leqslant 3$ we know already that there is only one smooth structure, by the theorems of Radó and Moise. When $n \geqslant 5$ we have:

Theorem 1.16 (Hirsch, Munkres, Stallings) *For $n \geqslant 5$, there is a unique smooth structure on $\mathbb{R}^n$. In other words, if M is a smooth n-manifold homeomorphic to $\mathbb{R}^n$, it is also diffeomorphic to $\mathbb{R}^n$.*

For $n = 4$ the situation is very different:

Theorem 1.17 (Freedman, Donaldson, Gompf, Taubes) *There are uncountably many mutually non-diffeomorphic exotic smooth structures on $\mathbb{R}^4$.*

(It was first shown by Freedman and Donaldson that there is at least one exotic $\mathbb{R}^4$ (by combining Freedman's techniques for topological 4-manifolds and Donaldson's techniques for smooth 4-manifolds). Later, Gompf showed that there are infinitely many exotic $\mathbb{R}^4$'s, and two years later Taubes showed that there are in fact uncountably many.)

A lemma that we will prove next week is the following, due to Mazur:

Lemma 1.18 *Let M be a closed, smooth, oriented, connected n-manifold. The the following are equivalent.*

- *There exists an n-manifold (also closed, smooth, oriented, connected) N such that $M \sharp N \cong \mathbb{S}^n$. In other words, the element $M \in \mathcal{M}_n$ is invertible.*
- *There is a smooth embedding of $M \smallsetminus$ point into $\mathbb{S}^n$.*
- *There is a diffeomorphism $M \smallsetminus$ point $\cong \mathbb{R}^n$.*

Using this, we can prove that, when $n \neq 4$, the submonoid $\mathcal{T}_n \subset \mathcal{M}_n$ is a group, and is moreover the maximal subgroup of $\mathcal{M}_n$ — in other words, an element of $\mathcal{M}_n$ is invertible if and only if it lies in $\mathcal{T}_n$. Suppose first that $M \in \mathcal{T}_n$, so M is homeomorphic to $\mathbb{S}^n$. Then $M \smallsetminus$ point is homeomorphic to $\mathbb{R}^n$, and therefore also diffeomorphic, by the theorem above. Then Mazur's lemma tells us that M is invertible. On the other hand, if we start by assuming that M is invertible, then Mazur's lemma tells us that $M \smallsetminus$ point is diffeomorphic to $\mathbb{R}^n$. Taking one-point compactifications we see that

$$M \cong_h (M \smallsetminus \text{point})^+ \cong_h (\mathbb{R}^n)^+ \cong_h \mathbb{S}^n$$

so $M \in \mathcal{T}_n$. (The notation $\cong$ by default means diffeomorphic in these notes, so $\cong_h$ is used for homeomorphisms.) The first homeomorphism follows because M is a compact topological space, so removing a point and then one-point compactifying does not change it up to homeomorphism (by uniqueness of the one-point compactification).

Using the topological Poincaré conjecture, we know that $\mathcal{T}_n = \mathcal{S}_n$, so (for $n \neq 4$) the monoid Θ_n is a quotient of the *group* $\mathcal{S}_n$, and must therefore also be a group. This proves Lemma 1.11 except when $n = 4$. However, this proof uses some large theorems (including Theorem 1.16 and, not least, the topological Poincaré conjecture!). We will give a more elementary proof that Θ_n is a group (for all n including $n = 4$) next week.

1.7. Outline

A rough outline of the rest of the lectures (becoming less detailed further into the future):

- We will fill in the details of this introduction and overview. In particular we will prove that $\sharp$ is a well-defined operation on Θ_n, so that it is a monoid; then we will prove that it is actually an abelian group. We will also show how the h-cobordism theorem implies the weak topological Poincaré conjecture in dimensions $\geqslant 5$ (using an additional fact in the case $n = 5$) and see some other consequences of the h-cobordism theorem. We will also define *twisted spheres* and give another interpretation (except for $n = 5$) of Θ_n as the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\mathbb{S}^{n-1}$ via the pseudoisotopy theorem.

- We will then go through some different methods of constructing exotic spheres, via twisted spheres (as in the previous point), links of singularities of complex varieties, Milnor's original construction via sphere bundles and plumbing.
- Then the main focus of the lectures will be to go through the proof of the theorem of Kervaire and Milnor [KM63] that Θ_n is finite for all $n \geqslant 4$, and do some explicit calculations along the way.
- Depending on how much time remains after this, we may look at the curvature of exotic spheres, look at dimension 4 in more detail, or ...

2. Lecture 2 – The isotopy extension theorem, the disc theorem and well-definedness of the connected sum.

I'll write the next three lectures up in more detail when I have time. For now, here is a rough summary of what was covered in the lectures.

- The space $C^\infty(M,N)$ of smooth maps $M \to N$ between two smooth manifolds, equipped with the C^∞ *compact-open topology*. Subspaces of embeddings and diffeomorphisms.
- The Theorem of Thom, Palais, Cerf and Lima that certain maps between embedding/diffeomorphism spaces are locally trivial fibre bundles (and hence Serre fibrations).
- Definitions of *isotopic* and *ambient isotopic* embeddings. Corollary: the isotopy extension theorem (and a parametrised version).
- The *Disc theorem* of Palais: any two smooth embeddings e_1, e_2 of the n-disc $\mathbb{D}^n$ into an n-dimensional, smooth, connected manifold M are isotopic (assuming additionally that e_1, e_2 are both orientation-preserving if M is orientable, otherwise no additional assumption) — and therefore ambient isotopic.
- A sketch of the proof, to see where the orientation-preserving assumption is needed — it's essentially because the general linear group $GL_n(\mathbb{R})$ has two path-components.
- Explicit definition of the connected sum $M\sharp N$ of two n-manifolds, depending on an auxiliary choice of embeddings $\mathbb{D}^n \hookrightarrow M$ and $\mathbb{D}^n \hookrightarrow N$ that preserve and reverse orientation respectively.
- Proposition: up to orientation-preserving diffeomorphism, the connected sum does not depend on this auxiliary choice. Thus we have a well-defined operation $\sharp\colon \mathcal{M}_n \times \mathcal{M}_n \longrightarrow \mathcal{M}_n$. It is easy to see that it is associative, commutative and that $M\sharp\mathbb{S}^n \cong M$ for all $M \in \mathcal{M}_n$, so this turns $\mathcal{M}_n$ into a commutative monoid.
- Lemma: The connected sum of two homotopy spheres is another homotopy sphere. Proof using the Hurewicz theorem, Whitehead theorem and Seifert-van-Kampen theorem, and via the notion of a *homology sphere*.
- Thus the subset $\mathcal{S}_n \subset \mathcal{M}_n$ of homotopy n-spheres is a submonoid.

3. Lecture 3 – Groups of manifolds under connected sum.

- Recollection of the precise definition of *cobordism* and *h-cobordism*. Easy exercise: the relation $\sim$ on n-manifolds of being h-cobordant (admitting an h-cobordism between them) induces an equivalence relation on $\mathcal{S}_n$.
- Lemma: the equivalence relation is compatible with the operation $\sharp$ on $\mathcal{S}_n$. More generally, if M, M', N, N' are any closed simply-connected n-manifolds and there are h-cobordisms $W\colon M \sim M'$ and $X\colon N \sim N'$, then there is also an h-cobordism $M\sharp N \sim M'\sharp N'$. (This was already mentioned as Lemma 1.9.) Proof using the *parametrised connected sum* of W and X along an embedded arc in each of them, plus the usual algebraic topology tools of Seifert-van-Kampen, Mayer-Vietoris and Whitehead's theorem.
- Note: the parametrised connected sum of two manifolds M and N along a common embedded submanifold $M \hookleftarrow L \hookrightarrow N$ is defined as long as the normal bundles of the two embeddings are isomorphic (and we choose such an isomorphism). In our case, L is an arc, which is contractible, so both normal bundles are automatically trivialisable.
- Corollary: the operation $\sharp$ descends to a well-defined operation on the quotient $\Theta_n = \mathcal{S}_n/\!\sim$.

Hence Θ_n is also a monoid, and the quotient map $q_n\colon \mathcal{S}_n \longrightarrow \Theta_n$ is a monoid homomorphism.

- Next we showed that Θ_n is actually an abelian group. We already know it's a commutative monoid, so we just have to show that every element admits an inverse.
- Step 1: If M is a simply-connected n-manifold, then $[M] = [\mathbb{S}^n] = 1 \in \mathcal{M}_n/{\sim}$ (in other words, M is h-cobordant to $\mathbb{S}^n$) if and only if M is diffeomorphic to the boundary of a compact, contractible $(n+1)$-manifold.
- (Proof by algebraic topology tools, including *Poincaré-Lefschetz duality*, which, together with the universal coefficient theorem, gives us the lemma that a simply-connected cobordism W, for which $\partial_0 W$ and $\partial_1 W$ are also simply-connected, is an h-cobordism if and only if $H_*(W, \partial_0 W) = 0$.)
- Step 2: If Σ is a homotopy n-sphere, then $\Sigma \sharp \overline{\Sigma}$ is diffeomorphic to the boundary of a compact, contractible $(n+1)$-manifold.
- Remark about how orientations are inherited by products and boundaries of manifolds. The conventions are such that, if M is oriented and you take the induced orientation on the product $M \times [0,1]$ and then the induced orientations on the two pieces $M \times \{0\}$ and $M \times \{1\}$ of its boundary, the obvious diffeomorphism $M \cong M \times \{0\}$ is orientation-preserving, whereas the obvious diffeomorphism $M \cong M \times \{1\}$ is orientation-*reversing*. See also footnote 4 on page 6.
- Then we went through the proof that $\mathcal{T}_n$ is the maximal subgroup of $\mathcal{M}_n$ (if $n \neq 4$), using Theorem 1.16 and the lemma of Mazur (Lemma 1.18).

4. Lecture 4 – Consequences of the h-cobordism theorem.

- Proof of Mazur's lemma, using the infinite connected sum trick (sometimes called *Mazur's swindle*).
- Note: we can extend $\mathcal{M}_n$ to a larger monoid by allowing non-orientable manifolds too. Take the set of all closed, connected, smooth n-manifolds and put an equivalence relation $\sim$ on them by saying that $M \sim N$ if and only if (a) at least one of M and N is non-orientable and they are diffeomorphic, or (b) both M and N are orientable and they have an orientation-preserving diffeomorphisms between them. The Disc theorem of Palais tells us that the operation of connected sum is well-defined on this larger set (and does not depend on any auxiliary choices), so we get a monoid $\mathcal{M}_n^{\pm}$ that contains $\mathcal{M}_n$ as a submonoid. Every element of $\mathcal{M}_n^{\pm} \smallsetminus \mathcal{M}_n$ is non-invertible, so $\mathcal{T}_n$ is also the maximal subgroup of $\mathcal{M}_n^{\pm}$.
- What is known about $\mathcal{M}_n$ and $\mathcal{M}_n^{\pm}$ for small n?
- By the classification of 1-manifolds, we have $\mathcal{M}_1^{\pm} = \mathcal{M}_1 = \{\mathbb{S}^1\}$ is the trivial monoid.
- By the classification of (compact) surfaces, we have $\mathcal{M}_2 \cong \mathbb{N}$, the free commutative monoid on one generator (the torus), and $\mathcal{M}_2^{\pm}$ is the commutative monoid given by the presentation $\langle T, P \mid TP = P^3 \rangle$.
- By the Kneser-Milnor prime decomposition theorem for closed, oriented, connected 3-manifolds, we have $\mathcal{M}_3 \cong \mathbb{N}^{\infty}$, the free commutative monoid on a countably infinite set of generators (the *prime* 3*-manifolds*). Note: Kneser proved in 1929 that every closed, oriented, connected 3-manifold has a decomposition into prime 3-manifolds, which implies that $\mathcal{M}_3$ is a *quotient of* the free commutative monoid generated by the set of prime 3-manifolds. In fact, now that the 3-dimensional Poincaré conjecture is known, this is easy to prove.[5] But it was not known until much later whether $\mathcal{M}_3$ is actually isomorphic to it. In 1962, Milnor showed that there are no relations between these generators, i.e., that the prime decomposition of a 3-manifold is unique up to homeomorphism (equivalently, since all topological 3-manifolds

[5] The argument is as follows. Try to decompose M as a connected sum of two other 3-manifolds, neither of them homeomorphic to $\mathbb{S}^3$. If you can't, then M is prime and you're done. If you can, apply the same procedure to the two factors. Now we need to show that this process terminates in a finite number of steps. Suppose for a contradiction that it does not. Let $m(G)$ denote the cardinality of a smallest generating set for a group G. By Seifert-van-Kampen's theorem, the fundamental group of a connected sum of 3-manifolds is the free product of their fundamental groups. Also, by a theorem of B. H. Neumann in 1943, we have $m(G * H) = m(G) + m(H)$ for any two groups G and H. Hence $m(\pi_1(M \sharp N)) = m(\pi_1(M)) + m(\pi_1(N))$. Since $m(\pi_1(M))$ was finite to begin with (because M is compact), we eventually, after a finite number of iterations of the process of decomposing into connected summands, end up with a connected summand N, not homeomorphic to $\mathbb{S}^3$, such that $m(\pi_1(N)) = 0$, in other words, N is simply-connected. But this is impossible, by the 3-dimensional Poincaré conjecture.

have a unique smooth structure, diffeomorphism) and permutation of the factors.

- The larger monoid $\mathcal{M}_3^{\pm}$ is actually not free, but it has a presentation similar to that of $\mathcal{M}_2^{\pm}$, except with infinitely many generators, namely $\langle P_i, Q_i \mid Q_i Q_0 = Q_i P_0 \rangle$, where the index i runs over all non-negative integers. The P_i are orientable and the Q_i non-orientable; the generators with positive subscript are *irreducible* (any embedded $\mathbb{S}^2$ is the boundary of an embedded $\mathbb{D}^3$); P_0 is $\mathbb{S}^1 \times \mathbb{S}^2$ and Q_0 is the (only) other $\mathbb{S}^2$ bundle over $\mathbb{S}^1$.
- As far as I know, little is known about the monoids $\mathcal{M}_n$ and $\mathcal{M}_n^{\pm}$ for $n \geqslant 4$. Up to dimension three there were no torsion elements, but in higher dimensions there is torsion, since for example $\mathbb{Z}/28 \cong \Theta_7 \cong \mathcal{S}_7 \subset \mathcal{M}_7$. For $n \geqslant 5$, the maximal subgroup of $\mathcal{M}_n$ (and of $\mathcal{M}_n^{\pm}$) is $\mathcal{T}_n = \mathcal{S}_n \cong \Theta_n$, which is finite, by the theorem of Kervaire and Milnor, and has been explicitly computed for $n \leqslant 64$. But not so much seems to be known about the non-invertible elements.
- Statement of the h-cobordism theorem and the s-cobordism theorem.
- Corollary: if two homotopy n-spheres with $n \geqslant 5$ are h-cobordant, then they are diffeomorphic. Thus $q_n \colon \mathcal{S}_n \longrightarrow \Theta_n$ is an isomorphism for $n \geqslant 5$.
- Definition of a *twisted sphere*. The subgroup Θ_n^{tw} of the group of homotopy spheres Θ_n.
- Corollary: for $n \geqslant 6$, every homotopy sphere is a twisted sphere, i.e., $\Theta_n^{\mathrm{tw}} = \Theta_n$. (This is actually true for $n \leqslant 5$ too, since in this case Θ_n is the trivial group – by Perelman for $n = 3$ and by Kervaire-Milnor for $n = 4, 5$.)
- Corollary: the weak topological Poincaré conjecture is true in dimensions $n \geqslant 5$, so we have $\mathcal{T}_n = \mathcal{S}_n$. For $n \geqslant 6$ this follows from the previous corollary, whereas for $n = 5$ it uses an additional fact from [KM63], which we will prove later in the course. In fact, for $n = 5$, using this additional fact and the h-cobordism theorem, we get the *smooth* Poincaré conjecture.
- Corollary: the h-cobordism theorem is also true for $n = 4$ if we make the extra assumption that $M \cong N \cong \mathbb{S}^4$. (This follows from the 5-dimensional smooth Poincaré conjecture (the previous point) together with the Disc theorem of Palais.)
- Note: the smooth h-cobordism theorem is *false* in general (where we only assume that M, N are simply-connected) for $n = 4$, as proved by Donaldson.
- Table summarising what is true, false and unknown in dimensions $n = 2, 3, 4$ and $\geqslant 5$ for the smooth/topological Poincaré conjectures and h-cobordism theorems.

5. Lecture 5 – Embeddings of topological manifolds; the pseudoisotopy theorem.

5.1. Embeddings of topological manifolds.

Collar and bicollar neighbourhood theorems.

- The *collar neighbourhood theorem* and the *bicollar neighbourhood theorem* for embeddings between smooth manifolds.
- Let M be a smooth, connected, compact manifold with boundary ∂M. Then there exists an embedding $c \colon \partial M \times [0,1) \hookrightarrow M$ such that $c(x,0) = x$ for all $x \in \partial M$ (a *collar neighbourhood* for ∂M). Moreover, the space of such embeddings (with the C^∞ compact-open topology) is contractible – in this sense, the choice of collar neighbourhood is 'unique'.
- Let M be a smooth, connected n-manifold and let $f \colon L \hookrightarrow M$ be a smooth embedding of a smooth, closed, connected $(n-1)$-manifold. Assume also that the embedding is *two-sided* in the sense that there exists a connected open neighbourhood U of $f(L)$ in M such that the complement $U - f(L)$ has two components. Then there exists a smooth embedding $g \colon L \times (-1,1) \hookrightarrow M$ extending f, i.e., such that $g(x,0) = f(x)$ for all $x \in L$. This is a *bicollar neighbourhood* for the embedding f. Again, the space (in the C^∞ compact-open topology) of bicollar neighbourhoods is contractible.
- Note: the hypotheses of connectedness and compactness are not really necessary. In the non-compact case we require the (bi)collar neighbourhoods to be *proper* embeddings.
- For topological manifolds and topological embeddings, the Collar Neighbourhood Theorem is also true (proved by M. Brown in 1962), whereas the Bicollar Neighbourhood Theorem is *false* in general. The famous Alexander Horned Sphere is a counterexample: it is an embedding $\mathbb{S}^2 \hookrightarrow \mathbb{S}^3$ that is two-sided but does not extend to an embedding of $\mathbb{S}^2 \times (-1,1)$.

Actually, it does extend 'on one side' but not on the other: it extends to $\mathbb{S}^2 \times [0, 1)$.

- More precisely, what Brown proved is that, in the setting of topological manifolds, if a collar neighbourhood exists *locally*, then it exists *globally*. Similarly, if a bicollar neighbourhood exists locally, then it exists globally. The difference arises in the fact that collar neighbourhoods always exist locally, simply by the definition of "topological manifold with boundary" – every point on the boundary has a coordinate chart that looks like $\mathbb{R}_+ \times \mathbb{R}^{n-1}$, giving a local collar neighbourhood. On the other hand, bicollar neighbourhoods of topological embeddings do *not* necessarily exist even locally. There is a point on the Alexander Horned Sphere such that no open neighbourhood of it admits a bicollar neighbourhood.
- A topological embedding $f\colon L \hookrightarrow M$ is called *locally flat* if for each point $x \in L$ there is a neighbourhood U of x in L and an embedding $U \times \mathbb{R}^{m-\ell} \hookrightarrow M$ that extends $f|_U$, where $\dim(M) = m$ and $\dim(L) = \ell$. If $m - \ell = 1$ then locally flat is equivalent to the existence of local bicollar neighbourhoods. Brown proved that a locally flat, codimension-one, two-sided embedding admits a bicollar neighbourhood. So the Collar Neighbourhood Theorem is true in the topological setting with no additional hypotheses, whereas the Bicollar Neighbourhood Theorem is true in the topological setting only once you add the hypothesis that the embedding is locally flat.
- Another example, which I didn't mention in the lecture, but which is illuminating, is the following. Take a non-trivial knot K in $\mathbb{S}^3$. The cone $C\mathbb{S}^3$ on $\mathbb{S}^3$ is homeomorphic to $\mathbb{D}^4$ and contains as a topological submanifold the cone CK on K, which is homeomorphic to $\mathbb{D}^2$. This embedding $\mathbb{D}^2 = CK \hookrightarrow C\mathbb{S}^3 = \mathbb{D}^4$ is not locally flat. In general, if $L \subset M$ is the image of a locally flat topological embedding, then for each point $x \in L$ there is a Euclidean ball neighbourhood B (the unit ball of a coordinate chart) in M centred at x such that the pair $(\partial B, \partial B \cap L)$ is homeomorphic to $(\mathbb{S}^{m-1}, \mathbb{S}^{\ell-1})$, where we are viewing $\mathbb{S}^{\ell-1}$ as the subspace $\mathbb{S}^{m-1} \cap \{(x_1, \ldots, x_m) \mid x_{\ell+1} = \cdots = x_m = 0\}$ of $\mathbb{S}^{m-1}$. However, for every Euclidean ball neighbourhood of the cone point in $C\mathbb{S}^3$, the intersection of its boundary with CK, as a pair, is homeomorphic to $(\mathbb{S}^3, K)$, which is not homeomorphic to $(\mathbb{S}^3, \mathbb{S}^1)$ since K is not the unknot.

The generalised Schönflies theorem.

- Alexander duality: if $A \subset \mathbb{S}^n$ is a non-empty, compact, locally contractible proper subset of the n-sphere, then there are isomorphisms $\widetilde{H}_k(\mathbb{S}^n - A) \cong \widetilde{H}^{n-k-1}(A)$.
- Fix $n \geqslant 2$ and a topological embedding $f\colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$. By Alexander duality we see that the complement of its image $\mathbb{S}^n - f(\mathbb{S}^{n-1})$ has two path-components, which we denote by U_1 and U_2.
- The generalised Schönflies theorem (due to Brown and independently Mazur and Morse) says that *if* the closure $\bar{U}_1$ of U_1 in $\mathbb{S}^n$ is, abstractly, a topological manifold with boundary, *then* it is in fact homeomorphic to the standard n-disc $\mathbb{D}^n$.
- Note: $\bar{U}_1$ and $\bar{U}_2$ are both topological manifolds with boundary if and only if f is a locally flat embedding if and only if f admits a bicollar neighbourhood.
- Corollary: if f is a locally flat embedding then there exists some self-homeomorphism $\phi\colon \mathbb{S}^n \longrightarrow \mathbb{S}^n$ such that the image of the embedding $\phi \circ f$ is the equator $\mathbb{S}^{n-1} \subset \mathbb{S}^n$.
- Note: it is not necessarily possible to find a ϕ such that $\phi \circ f$ is equal to the standard inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$ of the equator of $\mathbb{S}^n$ – it only follows that we can ensure that the *image* of $\phi \circ f$ is equal to the equator.
- Exercise: prove the *smooth Schönflies theorem* in dimensions $n \geqslant 5$: if $f\colon \mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$ is a smooth embedding, then there exists a self-diffeomorphicm $\phi\colon \mathbb{S}^n \longrightarrow \mathbb{S}^n$ such that the image of $\phi \circ f$ is the equator of $\mathbb{S}^n$. Moreover, we can choose ϕ to be isotopic to the identity, i.e., such that there is a path ϕ_t of self-diffeomorphisms with $\phi_1 = \phi$ and $\phi_0 = \mathrm{id}$. Thus f is ambient isotopic to an embedding onto the equator. Note: as before, we can only say that it is ambient isotopic to *an* embedding onto the equator, not *the* (standard) embedding onto the equator.
- Prove this, using Alexander duality, the Seifert-van-Kampen theorem, the Bicollar Neighbourhood theorem, the smooth h-cobordism theorem (Smale), the Disc theorem of Palais and the Isotopy Extension theorem.

The Annulus theorem.

- There is an analogous theorem to the generalised Schönflies theorem when we consider *two* disjoint embeddings $\mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$ simultaneously. Suppose f and g are two locally flat topological embeddings of $\mathbb{S}^{n-1}$ into $\mathbb{S}^n$ with disjoint images. By Alexander duality the complement $\mathbb{S}^n - (\mathrm{image}(f) \cup \mathrm{image}(g))$ has three path-components, which we denote by U_1, U_2 and U_3. It is easy to see that (up to reordering) we have $\bar{U}_1 = U_1 \cup \mathrm{image}(f)$, $\bar{U}_2 = U_2 \cup \mathrm{image}(g)$ and $\bar{U}_3 = U_3 \cup \mathrm{image}(f) \cup \mathrm{image}(g)$. The **Annulus Theorem** says that $\bar{U}_3$ is homeomorphic to $\mathbb{S}^{n-1} \times [0,1]$.
- Historically this was called the "Annulus Conjecture" (AC_n). It was proved by Radó for $n = 2$ in 1924, by Moise for $n = 3$ in 1952, by Kirby for all $n \geqslant 5$ in 1969 (several years *after* the generalised Schönflies theorem) and finally by Quinn for $n = 4$ in 1982.
- Relation to the Stable Homeomorphism Conjecture (SHC_n).
- Note: the Annulus theorem (plus the ubiquitous Alexander Trick) implies the generalised Schönflies theorem.

5.2. Implications between Poincaré conjectures and h-cobordism theorems.

- Reminder of the table from last week of the status (true; false; unknown) of the various smooth/topological Poincaré conjectures and h-cobordism theorems in different dimensions.
- Some implications:
  - smooth $\equiv$ topological in dimensions $\leqslant 3$ (Radó, Moise)
  - $\mathbf{sPC}_n$ and $\mathbf{sPC}_{n+1}$ implies $\mathbf{shCT}_{n+1}$ for $n \leqslant 3$ (using the Disc theorem of Palais)
  - $\mathbf{tPC}_n$ and $\mathbf{tPC}_{n+1}$ implies $\mathbf{thCT}_{n+1}$ for $n \leqslant 3$ (using the Annulus theorem)
  - $\mathbf{thCT}_{n+1}$ implies $\mathbf{tPC}_{n+1}$ (using the Alexander Trick)
  - $\mathbf{shCT}_{n+1}$ implies that every homotopy $(n+1)$-sphere is a *twisted* $(n+1)$-sphere
  - There are no (non-trivial) twisted 4-spheres (Cerf) [more on this in the next section]
- Then we went through the proof of the third implication, using the Annulus theorem from above.
- Remark: if W is a cobordism from M to N, the existence of a homeomorphism $W \cong M \times [0,1]$ automatically implies the existence of a homeomorphism $W \cong M \times [0,1]$ that restricts to the obvious map $W \supset M \longrightarrow M \times \{0\} \subset M \times [0,1]$. However, it does not in general imply the existence of a homeomorphism with "control" over both ends of the cobordism. On the other hand, if the topological group $\mathrm{Homeo}^+(M) = \mathrm{Homeo}^+(N)$ is path-connected, then it does imply this. The same remarks are also true for smooth manifolds and diffeomorphisms.

5.3. The pseudoisotopy theorem and twisted spheres.

- From now on we return to the world of smooth manifolds, smooth embeddings and diffeomorphisms.
- Recall: definition of a *twisted n-sphere*.
- Recall: definition of an *isotopy* between embeddings or diffeomorphisms.
- Definition: a *pseudoisotopy* between embeddings or diffeomorphisms.
- An easy exercise is to show that the relation on embeddings or diffeomorphisms of being pseudoisotopic (the existence of a pseudoisotopy between them) is an equivalence relation.
- A *pseudoisotopy diffeomorphism* of M is then defined to be a diffeomorphism $M \to M$ equipped with a pseudoisotopy to the identity. These form a topological group $C(M)$ and there is a continuous group homomorphism $C(M) \to \mathrm{Diff}(M)$ given by forgetting the pseudoisotopy. Note that its image is precisely the equivalence class of the identity under the relation of pseudoisotopy, which is a normal subgroup.
- The **Pseudoisotopy Theorem** of Cerf (1970) says that if M is a simply-connected manifold of dimension $\geqslant 5$ then $C(M)$ is path-connected.
- Corollary: under these hypotheses, if two diffeomorphisms $f, g \colon M \to M$ are pseudoisotopic, then they are isotopic. In other words, *a priori*, the equivalence relation of pseudoisotopy is weaker than the equivalence relation of isotopy; this theorem says that when $\pi_1(M) = 0$ and $\dim(M) \geqslant 5$ they are the same.
- Corollary: for $n \geqslant 6$, the topological group $\mathrm{Diff}^+(\mathbb{D}^n)$ is path-connected.

- Definition: $\mu\mathrm{Diff}(M)$ is the quotient of the group $\mathrm{Diff}(M)$ by the equivalence relation of pseudoisotopy. Equivalently, it is the quotient of $\mathrm{Diff}(M)$ by the normal subgroup $\mathrm{image}(C(M) \to \mathrm{Diff}(M))$. There is a surjecture group homomorphism $\pi_0\mathrm{Diff}(M) \twoheadrightarrow \mu\mathrm{Diff}(M)$.
- If M is orientable, we may similarly define $\mu\mathrm{Diff}^+(M)$ and there is a surjective group homomorphism $\pi_0\mathrm{Diff}^+(M) \twoheadrightarrow \mu\mathrm{Diff}^+(M)$.
- This fits into a certain long exact sequence. The restriction map $\mathrm{Diff}^+(\mathbb{D}^n) \to \mathrm{Diff}^+(\mathbb{S}^{n-1})$ is a locally trivial fibre bundle (cf. lecture 2) and the fibre over the identity is $\mathrm{Diff}_\partial(\mathbb{D}^n)$, the group of diffeomorphisms of $\mathbb{D}^n$ that restrict to the identity on the boundary. There is therefore a long exact sequence of homotopy groups (or sets) ending in

$$\cdots \to \pi_1\mathrm{Diff}^+(\mathbb{S}^{n-1}) \to \pi_0\mathrm{Diff}_\partial(\mathbb{D}^n) \to \pi_0\mathrm{Diff}^+(\mathbb{D}^n) \to \pi_0\mathrm{Diff}^+(\mathbb{S}^{n-1}) \tag{5.1}$$

where the last homomorphism is not necessarily surjective. Instead, one can show (exercise) that the exact sequence extends as follows:

$$\cdots \to \pi_0\mathrm{Diff}^+(\mathbb{D}^n) \to \pi_0\mathrm{Diff}^+(\mathbb{S}^{n-1}) \to \mu\mathrm{Diff}^+(\mathbb{S}^{n-1}) \to 0. \tag{5.2}$$

- From above we know that, for $n \geqslant 6$, the group $\pi_0\mathrm{Diff}^+(\mathbb{D}^n)$ is trivial, so from the exact sequence (5.2) we obtain an isomorphism $\pi_0\mathrm{Diff}^+(\mathbb{S}^{n-1}) \cong \mu\mathrm{Diff}^+(\mathbb{S}^{n-1})$. Alternatively, this isomorphism follows from the earlier corollary that the equivalence relations of isotopy and pseudoisotopy coincide for simply-connected manifolds of dimension at least 5.
- Side note: for homeomorphisms, the corresponding group is trivial. The Alexander Trick tells us that the homomorphism $\pi_0\mathrm{Homeo}^+(\mathbb{D}^n) \to \pi_0\mathrm{Homeo}^+(\mathbb{S}^{n-1})$ is an isomorphism, so in particular $\mu\mathrm{Homeo}^+(\mathbb{S}^{n-1}) = 0$.

6. Lecture 6 – Twisted spheres; Milnor's construction of exotic 7-spheres.

6.1. Twisted spheres (continued from lecture 5).

- The construction of twisted spheres is the following. Given an orientation-preserving diffeomorphism $f\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ we can glue two copies of the unit n-disc together along their boundaries using f. We denote the result by $\Sigma_f = \mathbb{D}^n \cup_f \mathbb{D}^n$. By the Alexander Trick, this is homeomorphic (but not necessarily diffeomorphic!) to $\mathbb{S}^n$. Thus we have a well-defined function

$$\Sigma_{(-)}\colon \mathrm{Diff}^+(\mathbb{S}^{n-1}) \longrightarrow \mathcal{T}_n \tag{6.1}$$

with codomain the set of smooth n-manifolds homeomorphic to the n-sphere. Note that $\mathbb{D}^n \cup_{\mathrm{id}} \mathbb{D}^n$ is diffeomorphic to $\mathbb{S}^n$. It is also not hard to write down a diffeomorphism between $\mathbb{D}^n \cup_{gf} \mathbb{D}^n$ and $\mathbb{D}^n \cup_g (\mathbb{S}^{n-1} \times [0,1]) \cup_f \mathbb{D}^n = \mathbb{D}^n \cup_g \mathbb{D}^n \sharp \mathbb{D}^n \cup_f \mathbb{D}^n$. Thus the function (6.1) is a monoid homomorphism. (If $n \neq 4$ we know that both sides are groups, in which case it is (automatically) a group homomorphism.)
- Lemma: the twisted sphere Σ_f depends only on the pseudoisotopy class of the diffeomorphism f. Equivalently, the homomorphism $\Sigma_{(-)}$ sends the normal subgroup $\mathrm{image}(C(\mathbb{S}^{n-1}) \to \mathrm{Diff}^+(\mathbb{S}^{n-1}))$ to the identity element $\mathbb{S}^n \in \mathcal{T}_n$. Thus the homomorphism (6.1) descends to a homomorphism

$$\Sigma_{(-)}\colon \mu\mathrm{Diff}^+(\mathbb{S}^{n-1}) \longrightarrow \mathcal{T}_n. \tag{6.2}$$

- Proof: by drawing pictures of diffeomorphisms between Σ_f, Σ_H and Σ_g when H is a pseudoisotopy between f and g. [I'll add these pictures later.]
- Lemma: the homomorphism (6.2) is injective.
- Sketch of proof: Suppose that $\Sigma_f \cong \mathbb{S}^n$. It will suffice to show that the diffeomorphism f of $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$ extends to a diffeomorphism of the whole disc $\mathbb{D}^n$, since the exact sequence (5.2) will then imply that $[f] = 0 \in \mu\mathrm{Diff}^+(\mathbb{S}^{n-1})$. Using the Disc theorem, we may find a self-diffeomorphism θ of $\mathbb{S}^n$ such that the composition

$$\mathbb{D}^n \hookrightarrow \mathbb{D}^n \cup_f \mathbb{D}^n = \Sigma_f \cong \mathbb{S}^n \xrightarrow{\theta} \mathbb{S}^n,$$

in which the left-hand map is the inclusion of the left-hand disc of $\mathbb{D}^n \cup_f \mathbb{D}^n$, is equal to the standard inclusion of the n-disc as the northern hemisphere of the n-sphere. It follows that the image of

$$\mathbb{D}^n \hookrightarrow \mathbb{D}^n \cup_f \mathbb{D}^n = \Sigma_f \cong \mathbb{S}^n \xrightarrow{\theta} \mathbb{S}^n, \tag{6.3}$$

in which the left-hand map is the inclusion of the *right*-hand disc of $\mathbb{D}^n \cup_f \mathbb{D}^n$, has image equal to the southern hemisphere of $\mathbb{S}^n$. Moreover, its restriction to $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$ is equal to $f \colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ followed by the inclusion of $\mathbb{S}^{n-1}$ as the equator of $\mathbb{S}^n$. Hence, after identifying the southern hemisphere of $\mathbb{S}^n$ with $\mathbb{D}^n$, the inclusion (6.3) gives us the desired extension of f to a diffeomorphism of the n-disc.

- Overall diagram:

$$\begin{array}{ccccccc} \pi_0\mathrm{Diff}^+(\mathbb{S}^{n-1}) & \overset{A}{\twoheadrightarrow} & \mu\mathrm{Diff}^+(\mathbb{S}^{n-1}) & \xrightarrow{(6.2)} & \mathcal{T}_n = \mathcal{S}_n & \overset{B}{\twoheadrightarrow} & \Theta_n \\ & & & \searrow\!\!\!\!\searrow & & & \uparrow \\ & & & & & & \Theta_n^{\mathrm{tw}} \end{array} \tag{6.4}$$

- The pseudoisotopy theorem implies that the homomorphism A is an isomorphism when $n \geqslant 6$.
- The h-cobordism theorem implies that the homomorphism B is an isomorphism when $n \geqslant 5$.
- We also saw earlier (via the h-cobordism theorem) that the subgroup Θ_n^{tw} is the whole group Θ_n when $n \geqslant 6$.
- Corollary: when $n \geqslant 6$ there is an isomorphism $\pi_0\mathrm{Diff}^+(\mathbb{S}^{n-1}) \cong \Theta_n$ given by sending an element $[f]$, where f is a diffeomorphism $\mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$, to the h-cobordism class of the n-manifold $\Sigma_f = \mathbb{D}^n \cup_f \mathbb{D}^n$.
- Definition: from now on, we will denote the group $\pi_0\mathrm{Diff}^+(\mathbb{S}^{n-1})$ by Γ_n.
- So the upshot of this section is that $\Gamma_n \cong \Theta_n$ for $n \geqslant 6$.
- What do we know about Γ_n in smaller dimensions?
- For $n = 5$, it will turn out (later in the course) that $\Theta_5 = 0$. On the other hand, nothing is known about the group $\Gamma_5 = \pi_0\mathrm{Diff}^+(\mathbb{S}^4)$.
- For $n = 1$, we obviously have $\mathrm{Diff}^+(\mathbb{S}^0) = \{\mathrm{id}\}$ and $\Theta_1 = \{\mathbb{S}^1\}$.
- For $n = 2$, it is an exercise to see that the inclusion $SO(2) \hookrightarrow \mathrm{Diff}^+(\mathbb{S}^1)$ is a homotopy equivalence. Since $SO(2)$ is connected, we have $\Gamma_2 = 0$. In this dimension we also know that $\Theta_2 = 0$, by the classification of surfaces.
- For $n = 3$, it is a theorem of Smale (1958) that the analogous statement is true in the next dimension up: the inclusion $SO(3) \hookrightarrow \mathrm{Diff}^+(\mathbb{S}^2)$ is a homotopy equivalence. Thus $\Gamma_3 = 0$. Also, we have $\Theta_3 = 0$ by Perelman's proof of the Poincaré conjecture.
- For $n = 4$, it is a theorem of Cerf (1969) that $\mathrm{Diff}^+(\mathbb{S}^3)$ is path-connected, i.e., that $\Gamma_4 = 0$, so there are no non-trivial twisted 4-spheres. This result was later (1983) strengthened by Hatcher to the fact that, as in the lower dimensions, the inclusion $SO(4) \hookrightarrow \mathrm{Diff}^+(\mathbb{S}^3)$ is a homotopy equivalence (this was known as the "Smale conjecture" until its proof by Hatcher).
- Note: in higher dimensions, the pattern does not continue. For example, $\Gamma_7 \cong \Theta_7$ is non-trivial (as we will show in the next section by constructing exotic 7-spheres), so $\mathrm{Diff}^+(\mathbb{S}^6)$ is not path-connected, whereas $SO(7)$ is.
- So in low dimensions $n \leqslant 3$ we have $\Gamma_n = \Theta_n = 0$, in high dimensions $n \geqslant 6$ we have $\Gamma_n \cong \Theta_n$ via the construction $\Sigma_{(-)}$ of twisted homotopy spheres above, and in between we have $\Gamma_4 = \Theta_5 = 0$, but nothing is known about Θ_4 or Γ_5. However, the vanishing of Γ_4 tells us that exotic 4-spheres, if they exist, must be impossible to construct as twisted spheres – in contrast to every other dimension $n \neq 4$, where every homotopy n-sphere is diffeomorphic to a twisted n-sphere.

6.2. Milnor's original 7-spheres.

- We will follow Milnor's original construction of exotic 7-spheres as the total space of a smooth bundle over $\mathbb{S}^4$ with fibre $\mathbb{S}^3$. Note that the standard 7-sphere is the total space of such a bundle: viewing $\mathbb{S}^3$ as the unit quaternions and $\mathbb{S}^7$ as the unit vectors in the 2-dimensional vector space $\mathbb{H}^2$, there is an action of $\mathbb{S}^3$ on $\mathbb{S}^7$ by quaternionic multiplication in each factor, with quotient $\mathbb{HP}^1 \cong \mathbb{S}^4$, and the quotient map $\mathbb{S}^7 \to \mathbb{S}^4$ is a smooth fibre bundle with fibre $\mathbb{S}^3$. The strategy is to classify all possible $\mathbb{S}^3$-bundles over $\mathbb{S}^4$, find out exactly which ones are *homeomorphic* to $\mathbb{S}^7$ and then show that some of these cannot be *diffeomorphic* to $\mathbb{S}^7$.

- Definition of a *principal G-bundle* for a topological group G.
- Classification of bundles with paracompact base B, structure group G and fibre F. Here G is a topological group and F is a G-space: the set of isomorphism classes of bundles of this form is in one-to-one correspondence with the set $[B, BG]$ of based homotopy classes of based maps from B to the classifying space BG of G. One direction of this correspondence goes as follows. Every topological group G has not only an associated classifying space BG, but also a principal G-bundle $\xi_G \colon EG \to BG$ over this space (this is a universal construction that depends only on G). Given a map $f \colon B \to BG$ we can take the pullback of ξ_G along f to obtain a principal G-bundle over B. The fibres of this bundle are G itself, but we may "replace the fibres" by any other (left) G-space to obtain another fibre bundle with structure group G (see the next bullet point). Doing this with the given G-space F we get a fibre bundle over B with fibre F and structure group G. Equivalently, we may *first* replace the fibres of ξ_G with the G-space F and *then* take the pullback of this fibre bundle along the map f – this gives something isomorphic to the result of the first procedure.
- Note about specifying a fibre bundle (with prescribed base, fibre and structure group) using a cocycle of transition functions (the fibre bundle construction theorem). This gives an easy way to define the "replacing the fibres" construction used in the previous point.
- Notation: given a principal G-bundle $\xi \colon P \to B$ and a G-space F, the result of replacing the fibres of ξ with F is denoted by $\xi \times_G F \colon P \times_G F \to B$. So the one-to-one correspondence described above sends a map $f \colon B \to BG$ to the fibre bundle $f^*(\xi_G) \times_G F \cong f^*(\xi_G \times_G F)$.
- In particular, if $F = N$ is a smooth manifold and G is its group $\mathrm{Diff}^+(N)$ of orientation-preserving diffeomorphisms, then this gives a classification of oriented manifold bundles over B with fibre N. Specifically, isomorphism classes of such manifold bundles are in one-to-one correspondence with $[B, B\mathrm{Diff}^+(N)]$.
- Note about a subtlety here. In the above, a "manifold bundle" means a fibre bundle with a *fibrewise* smooth structure, but not a *global* smooth structure. In other words, the fibres have compatible smooth structures, but these do not necessarily extend to a smooth structure on the total space of the bundle (and if they do, we do not make a choice of such an extension). Indeed, since B is any paracompact topological space, it does not make sense to speak of the total space being a smooth manifold.
- However, if $B = M$ is a smooth manifold, there is a notion of "smooth manifold bundle", where the total space has the structure of a smooth manifold and the projection is a smooth map. Isomorphism classes of oriented smooth manifold bundles over M with fibre N are *also* in one-to-one correspondence with the set of homotopy classes of maps $[M, B\mathrm{Diff}^+(N)]$. (Note: in this case, "isomorphism" means something stronger: it must be a *diffeomorphism* between the total spaces of the bundles that commutes with the two projections. For (fibrewise-smooth) manifold bundles, it just means a *homeomorphism* between the total spaces of the bundles that commutes with the two projections and that restricts to a diffeomorphism on each fibre.) The fact that both fibrewise-smooth and globally-smooth manifold bundles over M with fibre N are classified by the same thing amounts to the facts that (a) each (fibrewise-smooth) manifold bundle over M with fibre N is isomorphic (in the weaker sense) to a *smooth* manifold bundle over M with fibre N, and, moreover, if two *smooth* manifold bundles over M with fibre N are isomorphic in the weaker sense, then they are isomorphic in the stronger sense (the total spaces are diffeomorphic).
- Thus, whether we are talking about manifold bundles (which are just fibrewise-smooth) or *smooth* manifold bundles (where the total space is also equipped with a smooth structure) with base M and fibre N, in both cases the set of isomorphism classes is in one-to-one correspondence with the set $[M, B\mathrm{Diff}^+(N)]$, as long as we are careful to consider the correct notion of isomorphism in each case. (I have neglected to mention orientations in this remark, but the point that I am making is true either with or without orientations.)
- Specialising to the case $M = \mathbb{S}^4$ and $N = \mathbb{S}^3$ we see that oriented smooth manifold bundles over $\mathbb{S}^4$ with fibre $\mathbb{S}^3$ are in one-to-one correspondence with $[\mathbb{S}^4, B\mathrm{Diff}^+(\mathbb{S}^3)] = \pi_4(B\mathrm{Diff}^+(\mathbb{S}^3))$.
- Note about loop spaces: for any based space X the based loop space ΩX is defined to be

the space $\mathrm{Map}_*(\mathbb{S}^1, X)$ of based maps from $\mathbb{S}^1$ into X with the compact-open topology. Two general facts about this construction are that $\pi_k \Omega X \cong \pi_{k+1} X$ and, for a topological group G, there is a homotopy equivalence $\Omega BG \simeq G$. Using these facts, and Hatcher's theorem that the inclusion $SO(4) \hookrightarrow \mathrm{Diff}^+(\mathbb{S}^3)$ is a homotopy equivalence, we conclude that oriented smooth $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ are in one-to-one correspondence with

$$\pi_4(B\mathrm{Diff}^+(\mathbb{S}^3)) \cong \pi_3(\Omega B\mathrm{Diff}^+(\mathbb{S}^3)) \cong \pi_3(\mathrm{Diff}^+(\mathbb{S}^3)) \cong \pi_3(SO(4)). \tag{6.5}$$

- Note about universal covers of $SO(3)$ and $SO(4)$. First, $SO(3)$ is homeomorphic to $\mathbb{RP}^3$, so its universal cover is $\mathbb{S}^3$. Second, viewing $\mathbb{S}^3$ as the unit quaternions $\mathbb{S}^3 \subset \mathbb{H} = \mathbb{R}^4$, there is a continuous map
$$\mathbb{S}^3 \times \mathbb{S}^3 \longrightarrow \mathrm{Aut}_{\mathbb{R}}(\mathbb{H}) = GL_4(\mathbb{R})$$
sending a pair (x, y) to the $\mathbb{R}$-linear automorphism $z \mapsto xzy^{-1}$. Since x and y have unit norm, this automorphism preserves the norm of $\mathbb{H} = \mathbb{R}^4$, so the maps lands in $O(4)$. Since $\mathbb{S}^3 \times \mathbb{S}^3$ is connected, it must land in the connected component of the identity, namely $SO(4)$. It turns out that this map is a double cover (and thus the universal cover, since $\mathbb{S}^3 \times \mathbb{S}^3$ is simply-connected). Finally, after choosing basepoints, the inclusion $SO(3) \hookrightarrow SO(4)$ lifts to a unique map of universal covers $\mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$, and this map turns out to the the diagonal $x \mapsto (x, x)$.
- Since covering maps induce isomorphisms on higher homotopy groups, we see that oriented smooth $\mathbb{S}^3$-bundles over $\mathbb{S}^4$ are in one-to-one correspondence with (6.5) $\cong \pi_3(\mathbb{S}^3 \times \mathbb{S}^3) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- Definition: let $\xi_{i,j} \colon M_{i,j} \to \mathbb{S}^4$ denote the oriented smooth $\mathbb{S}^3$-bundle corresponding to the pair (i, j) under this isomorphism.
- We now have two questions to address:
(1) For which (i, j) is $M_{i,j}$ homeomorphic to $\mathbb{S}^7$?
(2) For which (i, j) is $M_{i,j}$ diffeomorphic to $\mathbb{S}^7$?
We will answer (1) completely, then construct an invariant λ of certain smooth 7-manifolds that vanishes on $\mathbb{S}^7$, and finally show that some of the $M_{i,j}$ that are homeomorphic to $\mathbb{S}^7$ have non-vanishing λ.
- The fibration sequence $\mathbb{S}^3 \to M_{i,j} \to \mathbb{S}^4$ induces a long exact sequence of homotopy groups, which immediately implies that $M_{i,j}$ must be at least 2-connected. Let $\delta_{i,j} \colon \pi_4\mathbb{S}^4 \to \pi_3\mathbb{S}^3$ denote the connecting homomorphism in this long exact sequence:
$$\cdots \to \pi_4 M_{i,j} \to \pi_4 \mathbb{S}^4 \xrightarrow{\delta_{i,j}} \pi_3 \mathbb{S}^3 \to \pi_3 M_{i,j} \to \cdots \tag{6.6}$$
- Lemma: $M_{i,j}$ is homeomorphic to $\mathbb{S}^7$ if and only if $\delta_{i,j}$ is an isomorphism.
- Note about classifying spaces of Lie groups and closed subgroups. If H is a closed subgroup of a Lie group G, then the projection map onto the coset space $G \to G/H$ is a principal H-bundle. Thus it corresponds to a (based homotopy class of a based) map $G/H \to BH$. Moreover, the inclusion induces a map of classifying space $BH \to BG$ and the sequence
$$H \hookrightarrow G \to G/H \to BH \to BG$$
is a *fibration sequence*. Also, there is a homotopy equivalence $BH \simeq EG \times_G (G/H)$ and under this equivalence the map $BH \to BG$ is the fibre bundle $\xi_G \times_G (G/H)$.
- Note about what exactly a *fibration sequence* means. Any continuous map $f \colon X \to Y$ between path-connected[6] spaces has a *homotopy fibre*, which is a space F_f together with a map $F_f \to X$, which are well-defined up to homotopy equivalence. Then the sequence $F_f \to X \to Y$ is called a *fibration sequence* and induces a long exact sequence on homotopy groups. A longer sequence $\cdots \to A \to B \to C \to D \to E \to \cdots$ is a *fibration sequence* if the homotopy fibre of each map $B \to C$ in the sequence is the previous map $A \to B$ in the sequence (up to homotopy equivalence) – in other words, if each subsequence of length 3 is a fibration sequence in the previous sense. Then any subsequence of length 3 induces the *same* long exact sequence on homotopy groups, up to the appropriate shift.

[6] For simplicity.

- Applying this fact to $G = SO(4)$ and its subgroup $H = SO(3)$ we get a fibration sequence

$$SO(3) \hookrightarrow SO(4) \to SO(4)/SO(3) \cong \mathbb{S}^3 \to BSO(3) \xrightarrow{h_3} BSO(4) \tag{6.7}$$

  and the bundle $\xi_{i,j}\colon M_{i,j} \to \mathbb{S}^4$ is the pullback of the $\mathbb{S}^3$-bundle h_3 along a certain map $\mathbb{S}^4 \to BSO(4)$ that we will denote by $f_{i,j}$. This gives us a map of long exact sequences of homotopy groups, which we used to prove:
- Lemma: the connecting homomorphism $\delta_{i,j}$ is an isomorphism if and only if $i - j = \pm 1$.
- Combining this with the previous lemma, we have a complete answer to question (1): the 7-manifold $M_{i,j}$ is homeomorphic to $\mathbb{S}^7$ if and only if $i - j = \pm 1$.
- Definition: Let k be an odd integer. We write $\xi_k\colon M_k \to \mathbb{S}^4$ to denote the smooth $SO(4)$-bundle $\xi_{i,j}\colon M_{i,j} \to \mathbb{S}^4$ with $i - j = 1$ and $i + j = k$. Each M_k is a smooth 7-manifold that is homeomorphic to $\mathbb{S}^7$. Moreover, it is also diffeomorphic to the boundary of a compact 8-manifold: namely the total space of the 4-disc bundle $\Xi_k\colon X_k \to \mathbb{S}^4$ obtained by taking the $SO(4)$-bundle $\xi_k\colon M_k \to \mathbb{S}^4$ and replacing the fibres $\mathbb{S}^3$ with the $SO(4)$-space $\mathbb{D}^4$.
- Recall the monoid $\mathcal{M}_7$ of closed, oriented, connected, smooth 7-manifolds under connected sum. Let $\mathcal{M}_7^\diamond$ be the submonoid of those 7-manifolds M with $H^3(M) = H^4(M) = 0$ and which are diffeomorphic to the boundary of a compact (oriented, connected, smooth) 8-manifold. Note: actually, by a theorem of Thom from 1954, every oriented 7-manifold is the boundary of an oriented 8-manifold, but we will not need to know this for our purposes, since all of the 7-manifolds that we need to consider (the M_k for k odd and $\mathbb{S}^7$) have an obvious 8-manifold of which they are the boundary (see the previous point), so

$$\mathbb{S}^7 \in \mathcal{M}_7^\diamond \qquad \text{and} \qquad M_k \in \mathcal{M}_7^\diamond \quad \text{for } k \in \mathbb{Z} \text{ odd.}$$

- The remaining steps are the following: we will define a function

$$\lambda\colon \mathcal{M}_7^\diamond \longrightarrow \mathbb{Z}/7\mathbb{Z} \tag{6.8}$$

  and then show that $\lambda(\mathbb{S}^7) = 0$ whereas $\lambda(M_k) = k^2 - 1$. It then follows that M_3, M_5 and M_7 are three distinct exotic 7-spheres, so $|\Theta_7| \geqslant 4$.
- Let $\overline{\mathcal{M}}_n$ denote the set of diffeomorphism classes of compact, connected, smooth, oriented n-manifolds with non-empty, connected boundary. The *boundary connected sum*, denoted by $\natural$, gives a well-defined monoid operation on $\overline{\mathcal{M}}_n$ in the same way that the connected sum $\sharp$ gives a monoid operation on $\mathcal{M}_n$. There is a monoid homomorphism

$$\partial\colon \overline{\mathcal{M}}_n \longrightarrow \mathcal{M}_{n-1} \tag{6.9}$$

  given by $M \mapsto \partial M$. There is also another interesting homomorphism

$$D\colon \overline{\mathcal{M}}_n \longrightarrow \mathcal{M}_n \tag{6.10}$$

  given by $M \mapsto M \cup_{\partial M} \bar{M}$, i.e., gluing two oppositely oriented copies of M along their common boundary. This is called the *double* of M. However, we will not need this homomorphism.
- Let $X \in \partial^{-1}\mathcal{M}_7^\diamond$. In other words, X is an 8-manifold in $\overline{\mathcal{M}}_8$ such that $\partial X \in \mathcal{M}_7^\diamond \subset \mathcal{M}_7$.
- Definition: the "Pontrjagin number" $q(X) \in \mathbb{Z}$. This definition also makes sense if $X \in \mathcal{M}_8$, in other words if X has empty boundary – we will need this extension of the definition later.
- Definition: the "signature" $\sigma(X) \in \mathbb{Z}$. Again, this definition also makes sense if $X \in \mathcal{M}_8$.
- Suppose that $X, Y \in \partial^{-1}\mathcal{M}_7^\diamond$ and $\partial X = \partial Y = M$. If we glue (with opposite orientations) along the common boundary, we obtain a closed, oriented 8-manifold $Z = X \cup_M \bar{Y} \in \mathcal{M}_8$.
- Lemma: the Pontrjagin numbers and signatures of X, Y and Z are related as follows:

$$q(Z) = q(X) - q(Y) \qquad \text{and} \qquad \sigma(Z) = \sigma(X) - \sigma(Y).$$

- The **Hirzebruch signature theorem** in dimension 8 tells us that, for a closed 8-manifold $Z \in \mathcal{M}_8$, the signature is determined by a certain linear combination of Pontrjagin numbers:

$$\sigma(Z) = \left(\tfrac{7}{45}p_2(Z) - \tfrac{1}{45}p_1(Z)^2\right) \cap [Z], \tag{6.11}$$

  where $[Z]$ is the fundamental class and $p_1(Z)$, $p_2(Z)$ are the first two Pontrjagin classes of (the tangent bundle of) Z.

- Corollary: for $X \in \partial^{-1}\mathcal{M}_7^\diamond$, the linear combination $2q(X) - \sigma(X)$ depends only on $\partial X \in \mathcal{M}_7^\diamond$ modulo 7. Thus we obtain a well-defined invariant of $M \in \mathcal{M}_7^\diamond$ by choosing a "coboundary" $X \in \overline{\mathcal{M}}_8$ with $\partial X = M$ and setting

$$\lambda(M) = 2q(X) - \sigma(X) \qquad (\text{mod } 7) \tag{6.12}$$

- Proof of the corollary: suppose that $X, Y \in \partial^{-1}\mathcal{M}_7^\diamond$ with $\partial X = \partial Y$. We just have to show that the following integer is zero modulo 7:

$$\begin{aligned}(2q(X) - \sigma(X)) - (2q(Y) - \sigma(Y)) &= 2q(Z) - \sigma(Z) \\ &= \tfrac{91}{45}q(Z) - \tfrac{7}{45}p_2(Z) \cap [Z] \\ &= \tfrac{7}{45}\big(13q(Z) - p_2(Z) \cap [Z]\big) \equiv 0 \qquad (\text{mod } 7)\end{aligned}$$

since 45 is invertible modulo 7.
- Calculations:
- It follows from the definitions that $q(\mathbb{D}^8) = \sigma(\mathbb{D}^8) = 0$ since $H_4(\mathbb{D}^8) = 0$. Thus $\lambda(\mathbb{S}^7) = 0$.
- Since $X_k \to \mathbb{S}^4$ is a fibre bundle with contractible fibre, it is a homotopy equivalence, and thus $H_4(X_k) \cong \mathbb{Z}$. The signature of X_k is therefore ± 1, depending on our choices of orientations, and we fix orientations such that $\sigma(X_k) = +1$. With more work (we didn't have time for this in the lecture) one can show that $q(X_k) = 4k^2$ with these orientations. Thus $\lambda(M_k) = k^2 - 1$.
- Note: we will see in the next lecture that $\lambda\colon \mathcal{M}_7^\diamond \to \mathbb{Z}/7\mathbb{Z}$ is not just a function, but a monoid homomorphism. Since $\lambda(M_3) = 1$ is a generator of $\mathbb{Z}/7\mathbb{Z}$, it is surjective, and it follows that there are at least six distinct exotic 7-spheres, namely

$$M_3,\ M_3\sharp M_3,\ M_3\sharp M_3\sharp M_3,\ \ldots\ldots,\ M_3\sharp M_3\sharp M_3\sharp M_3\sharp M_3\sharp M_3,$$

and $|\Theta_7| \geqslant 7$.

References

[KM63] Michel A. Kervaire and John W. Milnor. *Groups of homotopy spheres. I. Ann. of Math. (2)* 77 (1963), pp. 504–537 (↑ 4, 6, 8, 10).