Exotic spheres

Overview and lecture-by-lecture summary

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Abstract

This is a brief overview and a slightly less brief lecture-by-lecture summary of the topics covered in the course on Exotic Spheres that I taught in Bonn in the summer semester 2017 (starting on 25th April and ending on 19th July). See also the course webpage zatibq.com/exotic-spheres.

Overview

This is roughly the overview that I gave in the last 20 minutes of the final lecture on 19 July.

- Introduction
    - Monoids of manifolds: $\mathcal{M}_n$ and its submonoids.
    - Some useful theorems: including the Disc Theorem of Palais, the Isotopy Extension Theorem.
    - The group Θ_n of homotopy n-spheres up to h-cobordism; proof that it is a group.
    - The h-cobordism theorem and its corollaries.
    - Aside on topological embeddings of manifolds: the generalised Schönflies theorem and the Annulus theorem.
    - The Pseudoisotopy Theorem of Cerf; corollary that for $n \geqslant 6$ there is an isomorphism $\Theta_n \cong \pi_0 \mathrm{Diff}^+(\mathbb{S}^{n-1})$ via the twisted sphere construction.
- Milnor's original construction of exotic 7-spheres, via sphere bundles over spheres: classify the $\mathbb{S}^3$-bundles over $\mathbb{S}^4$, find out which ones are homeomorphic to $\mathbb{S}^7$, use an invariant λ (which we generalise below) to show that not all of these are diffeomorphic to $\mathbb{S}^7$.
- The general plumbing construction, with input a (bi-coloured) graph with vertices labelled by disc bundles over manifolds.
- Construction of invariant $\lambda\colon \mathcal{M}^{\diamond}_{4k-1} \longrightarrow \mathbb{Q}/\mathbb{Z}$, using Pontrjagin numbers, signature and the Hirzebruch Signature Theorem.
- Special case: fix the graph with two vertices and one edge – vary the two disc bundles. Condition (on one of the disc bundles) for when the boundary of the plumbing is a topological sphere. Computation of λ of these topological spheres.
- Another special case: label all vertices by a fixed thing, namely the unit disc subbundle of the tangent bundle of $\mathbb{S}^{2k}$, and see what happens as we vary the graph (which we require to be a tree). In particular we studied this for the E_8 tree.
- Construction of invariant $\lambda'\colon bP_{4k} \longrightarrow \mathbb{Z}/(\frac{\sigma_k}{8})\mathbb{Z}$.
  (This is surjective: if X is the plumbing of the E_8 tree, then $\sigma(X) = 8$ and so $\lambda'(\partial X) = 1$.)
- Construction of homotopy spheres as the link of an isolated singularity in a complex variety – Brieskorn spheres. (Important tool in the construction: the Milnor fibration.)
- Theorem: all homotopy spheres are stably parallelisable.
- Cobordism groups with (stable) tangential or normal structure; the Pontrajagin-Thom construction.
- Proof (Kervaire-Milnor) that Θ_n is finite for $n \geqslant 4$.
    - Step one: using the P-T construction to get a s.e.s. $0 \to bP_{n+1} \to \Theta_n \to (\text{finite}) \to 0$.
    - Surgery techniques — reduces the problem to trying to reduce $H_k(M)$ to zero by surgeries (where M is a $(k-1)$-connected parallelisable manifold of dimension $2k$ or $2k+1$ with homotopy sphere boundary).[1]
    - $n \equiv 0 \pmod 4$: This can always be done, by framed surgeries.

[1] Here we set $k = \lfloor \frac{n+1}{2} \rfloor$.

  - $n \equiv 2 \pmod 4$: This can also always be done, with more care and some subtleties about re-framing surgery data.
  - $n \equiv 3 \pmod 4$: This cannot always be done: the obstruction is the signature of M. But we can show that the invariant λ' is injective.
  - $n \equiv 1 \pmod 4$: This cannot always be done: the obstruction is the Kervaire (or Arf) invariant of M. But this lives in $\mathbb{Z}/2\mathbb{Z}$, and we constructed a surjection $\mathbb{Z}/2\mathbb{Z} \twoheadrightarrow bP_{n+1}$.
- Brief summary of what is known and unknown about the precise structure of the groups Θ_n.

Lecture-by-lecture summary

A more detailed summary of what was discussed in each lecture.

Lecture 1. (Tues 25 April)

- General introduction.
- Definitions of *topological sphere* and *homotopy sphere*.
- Connected sum and boundary connected sum.
- The monoids $\mathcal{M}_n \supseteq \mathcal{S}_n \supseteq \mathcal{T}_n$ of scco[2] n-manifolds, homotopy n-spheres and topological n-spheres respectively.
- Definition of *cobordism* and *h-cobordism*. The quotient monoid Θ_n of homotopy n-spheres up to h-cobordism.
- The various formulations of the Poincaré conjecture (smooth, topological, weak topological).
- What is known in dimensions $\leqslant 4$.
- The h-cobordism theorem in dimensions $\geqslant 5$.
- Corollary: the weak topological Poincaré conjecture, i.e., $\mathcal{T}_n = \mathcal{S}_n$, holds for $n \geqslant 6$.
- Corollary: h-cobordant homotopy spheres are diffeomorphic, i.e., the quotient $\mathcal{S}_n \to \Theta_n$ is an isomorphism, for $n \geqslant 5$.
- Exotic smooth structures on $\mathbb{R}^n$ (none for $n \neq 4$, uncountably many for $n = 4$).
- Mazur's lemma (a criterion for when an element of $\mathcal{M}_n$ is invertible).
- Corollary (also using the topological Poincaré conjecture and the uniqueness of smooth structures on $\mathbb{R}^n$ except for $n = 4$): $\mathcal{T}_n$ is a group, and therefore so is Θ_n (except possibly $n = 4$).

Lecture 2. (Wed 26 April)

- Spaces of embeddings and diffeomorphisms between manifolds.
- Isotopies and ambient isotopies; the Isotopy Extension Theorem (after Thom, Palais, Cerf and Lima).
- The Disc Theorem of Palais (sketch of proof).
- Definition of the connected sum; proposition: it does not depend on the auxiliary choices of embeddings of discs. The commutative monoid $\mathcal{M}_n$ of scco n-manifolds under connected sum.
- The connected sum of two homotopy spheres is a homotopy sphere. The connected sum of two topological spheres is a topological sphere.[3]
- Submonoids $\mathcal{T}_n \subseteq \mathcal{S}_n \subseteq \mathcal{M}_n$.

Lecture 3. (Tues 2 May)

- Cobordism and h-cobordism.
- The equivalence relation $\sim$ of being h-cobordant is compatible with the connected sum operation on $\mathcal{S}_n$ (proof using the *parametrised connected sum*). Therefore we have a well-defined quotient $\mathcal{S}_n \twoheadrightarrow \Theta_n \coloneqq \mathcal{S}_n/{\sim}$, and Θ_n is a commutative monoid.
- Proof that Θ_n is a group, with $[\Sigma]^{-1} = [\bar{\Sigma}]$.
- Proof that $\mathcal{T}_n$ is the maximal subgroup of $\mathcal{M}_n$ (if $n \neq 4$) using Mazur's lemma.

[2] smooth, connected, closed, oriented — although sometimes the second c will mean just compact, but possibly with non-empty boundary

[3] The second fact depends on the generalised Schönflies theorem, which comes later. (Thanks to Kaan Öcal for pointing out to me that this is less obvious than it seems.)

Lecture 4. (Wed 3 May)

- Proof of Mazur's lemma (with the infinite connected sum trick).
- We may extend $\mathcal{M}_n$ to a larger monoid $\mathcal{M}_n^{\pm}$ by allowing either non-orientable or orient*ed* manifolds.
- Structure of $\mathcal{M}_n$ and $\mathcal{M}_n^{\pm}$ for $n \leqslant 3$.
- The h-cobordism theorem and the s-cobordism theorem.
- Corollary: $q_n \colon \mathcal{S}_n \longrightarrow \Theta_n$ is an isomorphism for $n \geqslant 5$.
- Corollary: for $n \geqslant 6$, every homotopy sphere is a *twisted sphere*.[4]
- Corollary: the weak topological Poincaré conjecture is true, in other words $\mathcal{T}_n = \mathcal{S}_n$, in dimensions $n \geqslant 6$.[5]
- Summary of what is true, false and unknown for the Poincaré conjecture and h-cobordism theorem in their smooth and topological versions, in dimensions $n = 2, 3, 4$ and $\geqslant 5$.

Lecture 5. (Tues 9 May)

- The collar neighbourhood theorem and the bicollar neighbourhood theorem in the smooth setting.
- The collar neighbourhood theorem in the topological setting is also true, but the bicollar neighbourhood theorem fails topologically (the Alexander Horned Sphere).
- Example of a topological embedding that is not locally flat.
- Alexander duality.
- The generalised Schönflies theorem (for topological embeddings $\mathbb{S}^{n-1} \hookrightarrow \mathbb{S}^n$).
- The smooth Schönflies theorem in dimensions $n \geqslant 5$.
- The Annulus theorem — which implies the (topological) generalised Schönflies theorem.
- Proofs of some implications between different versions (smooth/topological, and different dimensions) of the Poincaré conjecture and the h-cobordism theorem. Tools: the Disc theorem of Palais, the Annulus theorem, the Alexander trick.
- Definition: *pseudoisotopy* and *pseudoisotopy diffeomorphism*.
- The *Pseudoisotopy Theorem* of Cerf.
- The group $\mu\mathrm{Diff}(M)$ of diffeomorphisms of M up to pseudoisotopy.
- Corollary: if M is simply-connected and $\dim(M) \geqslant 5$ then $\mu\mathrm{Diff}(M) \cong \pi_0\mathrm{Diff}(M)$. (And the same for orientation-preserving versions of these groups.)

Lecture 6. (Wed 10 May)

- Lemma: the construction of twisted spheres is an injective homomorphism $\mu\mathrm{Diff}^+(\mathbb{S}^{n-1}) \longrightarrow \mathcal{T}_n$.
- Together with the h-cobordism theorem and the pseudoisotopy theorem, this implies that we have an isomorphism $\pi_0\mathrm{Diff}^+(\mathbb{S}^{n-1}) \cong \Theta_n$ for $n \geqslant 6$.
- Summary of what is known in lower dimensions ($n \leqslant 5$) about $\Gamma_n \coloneqq \pi_0\mathrm{Diff}^+(\mathbb{S}^{n-1})$.
- Construction of Milnor's original exotic 7-spheres.
- Prerequisites: principal G-bundles, classification of principal G-bundles, manifold bundles (smooth vs. fibrewise smooth), replacing the fibres of a fibre bundle with a given structure group.
- First step: oriented smooth $\mathbb{S}^3$-bundles over $\mathbb{S}^4$, up to isomorphism, are in one-to-one correspondence with $\pi_3(SO(4)) \cong \pi_3(\mathbb{S}^3 \times \mathbb{S}^3) \cong \mathbb{Z}^2$. Denote them by $M_{i,j}$.
- Lemma: $M_{i,j}$ is homeomorphic to $\mathbb{S}^7$ if and only if $i - j = \pm 1$.
- For any odd k let M_k be $M_{i,j}$ with $i - j = 1$ and $i + j = k$.
- Construction of an invariant $\lambda(M) \in \mathbb{Z}/7\mathbb{Z}$ for any scco 7-manifold M with vanishing H^3 and H^4. This uses the Pontrjagin numbers and signature of an 8-manifold with boundary diffeomorphic to M, and the Hirzebruch signature theorem.
- Computations: $\lambda(\mathbb{S}^7) = 0$ whereas $\lambda(M_k) = k^2 - 1$. So there are at least 3 distinct exotic 7-spheres, namely M_3, M_5 and M_7. In fact it turns out that λ is a homomorphism, so there are at least 6 distinct exotic 7-spheres.

[4] This is in fact true for all n, since Θ_n is trivial for $n \leqslant 5$, by classification for $n = 1, 2$, by Perelman for $n = 3$ and by Kervaire-Milnor for $n = 4, 5$.

[5] This is also true for all n.

Lecture 7. (Wed 24 May)

- Comparision between construction of sphere bundles over spheres (via clutching functions) and the plumbing construction (in a special case).
- The idea of the general plumbing construction (details next lecture).
- Pontrjagin classes and numbers, for closed manifolds and more generally for compact manifolds with certain conditions on the boundary (e.g. homology sphere boundary).
- The Hirzebruch Signature Theorem.
- Lemmas about how Pontrjagin numbers and signature behave under (a) gluing along common boundary and (b) boundary connected sum (i.e. gluing along a disc in each boundary).
- Definition of the invariant λ.
- Corollaries: this invariant is (a) well-defined and (b) a homomorphism $\mathcal{M}_{4k-1}^{\diamond} \longrightarrow \mathbb{Q}/\mathbb{Z}$.

Lecture 8. (Tues 30 May)

- Details of the general plumbing construction (unoriented and oriented versions).
- Lemma: the plumbing construction depends only on the input graph and the disc bundles that label its vertices, and not on the extra auxiliary choices that we made (certain embeddings of discs and trivialisations of the bundles over those discs). This is analogous to the fact that the connected sum is independent of the additional choice of (suitably-oriented) embedded discs.
- Homotopy type of the plumbing, operations that correspond to boundary connected sum of plumbings and to fibrewise connected sum of disc bundles.
- Alternative, explicit description of the plumbing construction in the special case of a graph with two vertices (plumbing together two disc bundles over spheres).
- Let X be the plumbing of two disc bundles over spheres, described by clutching functions f and g. Lemma: condition on f that implies that ∂X is a topological sphere.

Lecture 9. (Wed 31 May)

- Calculation that X has signature zero (under certain conditions on f and g).
- The Pontrjagin homomorphism.
- Calculation of the Pontrjagin numbers of X, and therefore of $\lambda(\partial X)$, in terms of the Pontrjagin homomorphism evaluated on f and g.
- Corollary (using also a theorem of Bott and Milnor): a lower bound on the number of exotic spheres in dimension $4k-1$.

Lecture 10. (Tues 13 June)

- (Clarification about the signature of a manifold, and the calculation of $\sigma(X)$ from last lecture.)
- Study the plumbing construction on a tree Γ with vertices labelled by a disc bundle over a sphere (described by clutching functions). Definition: the intersection matrix of such a labelled tree.
- The Euler homomorphism.
- Lemma: the signature of $P\Gamma$ depends only on the intersection matrix of Γ.

Lecture 11. (Wed 14 June)

- Lemma: condition for when $\partial P\Gamma$ is a topological sphere (the intersection matrix of Γ is non-singular over $\mathbb{Z}$).
- Specialise to the E_8 tree with every vertex labelled by $T_1\mathbb{S}^{2k}$: the intersection matrix is non-singular and its signature is 8.
- Corollary: a lower bound on the number of exotic spheres that bound a parallelisable manifold in dimension $4k-1$.
- Philosophy of how to define invariants of homotopy spheres Σ: not by using invariants of their tangent bundles (this cannot work), but by using invariants of (the tangent bundles of) all possible smooth manifolds whose boundary is Σ.
- Definition of the invariant $\lambda' \colon bP_{4k} \longrightarrow \mathbb{Z}/(\frac{\sigma_k}{8})\mathbb{Z}$. Proof that it is a well-defined homomorphism.

- Remark about how λ and λ' are related.

Lecture 12. (Tues 20 June)

- Complex varieties, singular points and their links.
- The Milnor Fibration Theorem.
- Some lemmas about when vector bundles are (stably) trivialisable.
- Corollary: the link of an isolated singularity bounds a parallelisable manifold.
- Setup for constructing Brieskorn spheres Σ_a, depending on a tuple of integers a.
- Note about twisted homology and the Serre spectral sequence.
- Aside: some computations of Phạm.
- Corollary (Theorem of Brieskorn): a combinatorial condition on a that ensures that Σ_a is a topological sphere.
- Corollary (Theorem of Brieskorn): a combinatorial formula for the signature of the (canonical) parallelisable manifold that Σ_a bounds.
- Special case: $a = (6\ell - 1, 3, 2, \ldots, 2)$.

Lecture 13. (Wed 21 June)

- The J-homomorphism.
- Obstruction theory for finding a section of a bundle; relation to Pontrjagin classes and to the J-homomorphism.
- Proof that all homotopy spheres are stably parallelisable.
- Aside: Stiefel-Whitney classes and numbers.
- [Thom]: two unoriented manifolds are cobordant if and only if their SW numbers agree.
- [Wall, Milnor]: two oriented manifolds are oriented-cobordant if and only if their SW and P numbers agree.
- Corollary: every homotopy sphere is the oriented boundary of some (compact!) oriented manifold.

Lecture 14. (Tues 27 June)

- Transversality; Thom's transversality theorem.
- The smooth approximation theorem.
- The tubular neighbourhood theorem.
- The oriented cobordism group Ω_n.
- Lemma: the canonical homomorphism $\Theta_n \longrightarrow \Omega_n$ is zero.
- Variations: unoriented, spin, framed cobordism groups. In general: cobordism groups of manifolds with a tangential structure.
- Other variations: manifolds with a stable tangential structure, embedded manifolds with a normal structure.

Lecture 15. (Wed 28 June)

- More precise definition/construction of cobordism groups of manifolds with a tangential structure.
- Aside about stable almost complex structures.
- Lemma: relation between cobordism groups of manifolds with a stable framing and cobordism groups of embedded manifolds with a normal framing.
- The Pontrjagin-Thom construction (and theorem).
- (Aside about Thom spectra and Madsen-Tillmann spectra.)

Lecture 16. (Thu 29 June)

- How Pontrjagin (and Rokhlin) used the PT construction to compute the (low-degree) stable homotopy groups of the sphere spectrum.
- Construction of a homomorphism from Θ_n to a certain quotient of π_n of the sphere spectrum.
- Lemma: its kernal is precisely $bP_{n+1} \leqslant \Theta_n$.
- Corollary: Θ_n is finite if and only if bP_{n+1} is finite.
- Brief look ahead to how we will prove that bP_{n+1} is finite, using surgery.

- Definition: (framed) surgery datum λ, the effect M_λ of surgery on M and the associated cobordism W_λ.
- Lemma: the effect of surgery in low dimensions.
- Corollary: to finish the proof, it suffices to reduce $H_k(M)$ to zero by surgeries.

Lecture 17. (Tues 4 July)

- Case $n \equiv 0 \pmod 4$.
- Lemma: the effect of surgery just below the middle dimension.
- How to remove a $\mathbb{Z}$ summand from $H_k(M)$.
- How to decrease the size of the torsion subgroup of $H_k(M)$.
- Corollary: we may always reduce $H_k(M)$ to zero in this case, so Θ_{4k} is finite.

Lecture 18. (Wed 5 July)

- Case $n \equiv 2 \pmod 4$.
- Re-framing a framed surgery datum.
- The linking number $L(-,-)$ of two torsion homology classes in a manifold.
- As yesterday, we may always reduce $H_k(M)$ to a finite abelian group, so we assume this from now on.
- Most of the proof of a technical lemma: if ℓ'/ℓ is not an integer, where ℓ is the order of λ and ℓ' is the order of the dual surgery datum λ', then performing λ-surgery reduces the size of $H_k(-)$, after possibly re-framing λ.

Lecture 19. (Tues 18 July)

- Finishing case $n \equiv 2 \pmod 4$.
- Lemma: $\ell'/\ell = \pm L(\lambda, \lambda)$ modulo 1.
- Corollary: we may always reduce the size of $H_k(M)$ by either one or two sugeries, as long as this group contains an element with non-trivial self-linking number.
- Lemma: if all elements have trivial self-linking number, then $H_k(M)$ is 2-torsion.
- Corollary: in this case, we may always perform a surgery so that either (i) the torsion subgroup of $H_k(-)$ is smaller or (ii) it is the same size, but *not* all 2-torsion.
- Corollary: thus we may always reduce $H_k(M)$ to the trivial group, so Θ_{4k+2} is finite.

Lecture 20. (Wed 19 July)

- Case $n \equiv 1$ or $3 \pmod 4$.
- Explanation about why this case is harder: this time we have to perform surgeries in the middle dimension, not only just below the middle dimension. Not every homology class may be represented by a surgery datum.
- Key lemma: two conditions (a) and (b) that ensure that we may reduce $H_k(M)$ to the trivial group (which implies that $[\partial M] = 0 \in bP_{2k}$).
- Case $n \equiv 3 \pmod 4$.
- Condition (a) implies condition (b) in this case, so we just have to study condition (a). Question: when does $H_k(M)$ have a basis such that the intersection form has a certain matrix with respect to it?
- Answer: if and only if the signature of M is zero.
- Corollary: the invariant λ' introduced earlier is an isomorphism $bP_{4k} \longrightarrow \mathbb{Z}/(\frac{\sigma_k}{8})\mathbb{Z}$.
- Hence bP_{4k} is finite and therefore so is Θ_{4k-1}.
- Case $n \equiv 1 \pmod 4$.
- Condition (a) is automatic in this case, but condition (b) is not. Question: when does an embedded $\mathbb{S}^k$ in M^{2k} have a trivial normal bundle?
- The answer depends on a certain invariant ψ, which may be used to define the Kervaire (or Arf) invariant $c(M) \in \mathbb{Z}/2\mathbb{Z}$ of M.
- Corollary: if $c(M)$ is zero then $[\partial M] = 0 \in bP_{2k}$.
- Corollary: bP_{2k} is either 0 or $\mathbb{Z}/2\mathbb{Z}$ when k is odd. Thus Θ_{4k+1} is finite.
- Brief overview of what is known and unknown related to the precise structure of the groups Θ_n.