Exercise sheet 3

Due before the lecture on Monday, 5 November 2018.

Exercise 1. (5 points) Let X be a topological space and write

$$\Delta = \{(x,x) \mid x \in X\} \subseteq X \times X$$

for the diagonal.

(a) Show that X is Hausdorff if and only if $\Delta \subseteq X \times X$ is closed with respect to the product topology.

(b) Assume that X is compactly generated. Show that X is weakly Hausdorff if and only if $\Delta \subseteq X \times X$ is closed with respect to the compactly generated product topology, i.e. the k-ification of the usual product topology.

Exercise 2. (4 points) Consider the trivial fibre bundle $I \times I \to I$. Find a subspace $E \subseteq I \times I$ such that the projection restricts to a fibration $E \to I$ that is not a fibre bundle.

Exercise 3. (5 points)

(a) Show that $\pi_n(S^n) \cong \mathbb{Z}$.

(b) Assume that $S^m \to S^n$ is a fibre bundle with fibre S^k. Show that $k = n-1$ and $m = 2n-1$.

(c) Let $E \to B$ be a fibration over a path-connected space B such that the inclusion of the fibre $F \to E$ is null-homotopic. Construct isomorphisms $\pi_n(B) \cong \pi_n(E) \times \pi_{n-1}(F)$. Deduce that $\pi_7(S^4)$ and $\pi_{15}(S^8)$ contain $\mathbb{Z}$ as a summand. (You may use without proof the fact that there exist fibre bundles as in (b) for $n = 4$ and $n = 8$.)

Exercise 4. (6 points) Let $A \to X$ be a cofibration of spaces and let f and g be homotopic maps $X \to Y$ that agree on A.

(a) Show that if A is contractible, then f and g are homotopic relative to A. (*Hint:* Fix a homotopy $h \colon f \simeq g$. Choose a null-homotopy k of $f\big|_A$ and use it to construct a homotopy of homotopies $K \colon A \times I \times I \to Y$ between $h\big|_A$ and the constant homotopy $f\big|_A \simeq g\big|_A$. Then apply the homotopy extension property of $A \times I \to X \times I$.)

(b) Show that the contractibility assumption in (a) is necessary. (*Hint:* Consider maps of pairs $(D^1, S^0) \to (S^1, *)$.)