Exercise sheet 4

Due before the lecture on Monday, 12 November 2018.

Exercise 1. (5 points) Let $n \geqslant 2$ and let X be a homology n-sphere, i.e. a space such that

$$\widetilde{H}_i(X;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$$

Show: if X is a simply-connected CW complex, then it is homotopy equivalent to S^n.

Exercise 2. (5 points) Compute all of the homotopy groups of $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$. (*Hint:* consider suitable fibre bundles $S^\infty \cong \operatorname{colim}_n S^n \to \mathbb{R}P^\infty$ respectively $S^\infty \cong \operatorname{colim}_n S^{2n+1} \to \mathbb{C}P^\infty$ and use Exercise 4b.)

Exercise 3. (4 points) Consider the trivial fibre bundle $I \times I \to I$. Find a subspace $E \subseteq I \times I$ such that the projection restricts to a quasi-fibration $E \to I$ that is not a Serre fibration.

Exercise 4. (6 points) Let $X_0 \to X_1 \to X_2 \to \cdots$ be a countably infinite sequence of closed inclusions of T_1 spaces and write $X = \operatorname{colim}_i X_i$ for the colimit of this sequence. (Recall that a space is T_1 if points are closed.)

(a) Show that the canonical map

$$\operatorname*{colim}_i \pi_n(X_i) \longrightarrow \pi_n(X)$$

induced by the inclusions $X_i \to X$ is a bijection for all n. (*Hint:* show that if K is compact, then the image of any map $K \to X$ must be contained in some X_i.)

(b) Deduce that $S^\infty \cong \operatorname{colim}_i S^i$ is contractible.

(c) Assume that in the following commutative diagram of T_1 spaces, all horizontal maps are closed inclusions and all vertical maps are weak equivalences.

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots \end{array}$$

Show that the induced map

$$f\colon X = \operatorname*{colim}_i X_i \longrightarrow \operatorname*{colim}_i Y_i = Y$$

is a weak equivalence.

Remark: One can show that weak Hausdorff spaces are T_1 and that sequential colimits along closed inclusions in the category of compactly generated weak Hausdorff spaces can be computed in the category of all topological spaces. Consequently, the above statements also hold in the "convenient category of spaces".