Exercise sheet 5

Due before the lecture on Monday, 19 November 2018.

Exercise 1. (Left properness) (5 points) For a pushout

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ B & \longrightarrow & D \end{array}$$

in the category CGWH show the following:

(a) If i is a cofibration, then so is i'.

(b) If i is a cofibration and a homotopy equivalence, then so is i'.

You may use without proof the fact that pushouts in CGWH with one "leg" a cofibration can be computed in Top.
(*Hint:* For (b), first show that i is the inclusion of a deformation retract, i.e. there exists $r\colon B \to A$ such that $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_B$ rel. A.)

Exercise 2. (5 points) Let $\eta\colon S^3 \to S^2$ denote the Hopf fibration and consider the sequence of suspension homomorphisms

$$\pi_1(S^0) \xrightarrow{S} \pi_2(S^1) \xrightarrow{S} \pi_3(S^2) \xrightarrow{S} \pi_4(S^3) \xrightarrow{S} \pi_5(S^4) \xrightarrow{S} \cdots$$

(a) Show that the sequence stabilizes at $\pi_4(S^3)$ and that the latter group is generated by $S([\eta])$.

(b) Show that $2 \cdot S([\eta]) = 0$ in $\pi_4(S^3)$. Deduce that this group is either trivial or has two elements. (*Hint:* First show that $\eta \circ c = r \circ \eta$ where c is complex conjugation in both coordinates of unit vectors in $\mathbb{C}^2$ and r is a suitable reflection of S^2.)

Remark: Indeed $\pi_4(S^3) \cong \mathbb{Z}/2$, but we don't have the technology available to show that $S([\eta]) \neq 0$ yet. This is usually done via cohomology operations.

Exercise 3. (5 points) Let $\mathbb{Q}$ be the rational numbers equipped with the subspace topology of $\mathbb{R}$. Topologise $X = \mathbb{Q} \cup \{\infty\}$ such that $U \subseteq X$ is open if and only if either $\infty \notin U$ and U is open in $\mathbb{Q}$, or $\infty \in U$ and $X - U$ is compact. Show that X is a weak Hausdorff space that is not Hausdorff.

Exercise 4. (5 points) For maps of based spaces $f\colon S^i \to X$ and $g\colon S^j \to X$, define their *Whitehead product* $[f,g]$ to be the composition

$$[f,g] : S^{i+j-1} \longrightarrow S^i \vee S^j \xrightarrow{f \vee g} X,$$

where the first map is the attaching map of the top-dimensional cell of $S^i \times S^j$ with its standard cell structure.

(a) Show that this construction gives rise to a well-defined operation

$$[- \, , -] : \pi_i(X) \times \pi_j(X) \longrightarrow \pi_{i+j-1}(X),$$

which is bilinear when $i, j \geqslant 2$.

(b) Identify the Whitehead product for $i = j = 1$ and justify the square bracket notation.