Exercise sheet 7

Due before the lecture on Monday, 3 December 2018.

For the first two exercises, call X an *Eilenberg-MacLane space of type* $K(G,n)$ if $\pi_n(X) \cong G$ and all other homotopy groups of X vanish. This is equivalent to the definition (via the representability theorem of E. Brown) given in lectures, as can be seen by combining Exercise 2 with the calculation of homotopy groups discussed in the lectures.

Exercise 1. (5 points) Let $n \geqslant 2$ and let G be an abelian group with presentation

$$0 \to \bigoplus_{j\in J} \mathbb{Z} \xrightarrow{\varphi} \bigoplus_{i\in I} \mathbb{Z} \longrightarrow G \to 0.$$

(a) Choose a map

$$f\colon \bigvee_{j\in J} S^n \longrightarrow \bigvee_{i\in I} S^n$$

so that $H_n(f;\mathbb{Z}) = \varphi$ under a chosen identification $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$. Show that the homotopy cofibre $X = C_f$ of f is $(n-1)$-connected and satisfies $\pi_n(X) \cong G$.

(b) Construct an Eilenberg-MacLane space of type $K(G,n)$ by attaching cells of dimension at least $n+2$ to X.

Exercise 2. (5 points) Assume that K and K' are CW-complexes that are both Eilenberg-MacLane spaces of type $K(G,n)$, $n \geqslant 2$. Show that K and K' are homotopy equivalent. (*Hint:* If K is constructed as in Exercise 1, find a map $g\colon X \to K'$ that induces an isomorphism on π_n and show that g extends to all of K.)

Exercise 3. (5 points) Let $X = \mathrm{colim}_n(X^n)$ be a based CW-complex, written as the colimit of its skeleta $X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow X^3 \hookrightarrow \cdots$, and let Y be any based space. Consider the sequence of based sets

$$* \to Ph(X,Y) \longrightarrow \langle X,Y\rangle \longrightarrow \lim_n \langle X^n, Y\rangle \to *, \tag{1}$$

where $Ph(X,Y) \subseteq \langle X,Y\rangle$ is the subset consisting of based homotopy classes of phantom maps and the second map is induced by the inclusions $i_n\colon X^n \hookrightarrow X$. Show that (1) is an exact sequence of based sets, and show that it is an exact sequence of groups if X is a suspension.

Exercise 4. (5 points) Let $\theta\colon T \to U$ be a natural transformation between half-exact contravariant functors defined on the category CW_* of based, connected CW-complexes and taking values in the category of abelian groups.[1]

(a) For a fixed $n > 0$, assume that the function $\theta(S^m)\colon T(S^m) \to U(S^m)$ is bijective for all $m < n$ and surjective for $m = n$. Show that, for every based, connected, finite-dimensional CW-complex X, the function

$$\theta(X)\colon T(X) \longrightarrow U(X) \tag{2}$$

is bijective for $\dim(X) < n$ and surjective for $\dim(X) = n$.

(b) Now assume that the function $\theta(S^m)\colon T(S^m) \to U(S^m)$ is bijective for all $m > 0$. Show that the function (2) is bijective for every based, connected CW-complex X.

[1] The following statements are also true more generally when T and U take values in the category of sets, but the proof of part (a) in this more general setting is much more involved.