Exercise sheet 8

Due before the lecture on Monday, 10 December 2018.

Exercise 1. (6 points) For a based space (Y, y_0), let ΩY denote its based loop space equipped with the usual H-space structure given by concatenation of based loops. Define the *Moore loop space* of Y to be the quotient space

$$\Omega' Y = (\mathbb{R}_{\geqslant 0} \times \Omega Y)/(\{0\} \times \Omega Y).$$

It is a (strictly associative and strictly unital) topological monoid with multiplication $\Omega' Y \times \Omega' Y \to \Omega' Y$ given by

$$[t_1, \gamma_1] \cdot [t_2, \gamma_2] = [t_1 + t_2, \gamma]$$

where γ is the based loop

$$\gamma(s) = \begin{cases} \gamma_1\big(\frac{s(t_1+t_2)}{t_1}\big) & 0 \leqslant s \leqslant \frac{t_1}{t_1+t_2} \\ \gamma_2\big(\frac{s(t_1+t_2)-t_1}{t_2}\big) & \frac{t_1}{t_1+t_2} \leqslant s \leqslant 1. \end{cases}$$

(These statements may be assumed without proof.)

(a) Show that the map

$$\lambda\colon \Omega Y \longrightarrow \Omega' Y, \;\; \gamma \longmapsto [1, \gamma]$$

is a homotopy equivalence. Is it a map of H-spaces?

Now let (X, e) be a based space. For $n \geqslant 2$, the n-fold *reduced product* is the quotient space

$$J_n(X) = X^n / \sim$$

by the equivalence relation generated by

$$(x_1, \ldots, x_i, e, \ldots, x_n) \sim (x_1, \ldots, e, x_i, \ldots, x_n)$$

for all $1 \leqslant i \leqslant n-1$. We set $J_1(X) = X$. For each $n \geqslant 1$, there is an inclusion $J_n(X) \hookrightarrow J_{n+1}(X)$ given by inserting a basepoint in one of the coordinates. We call

$$J(X) = \operatorname*{colim}_n J_n(X)$$

the *James construction* on X.

(b) Assume that X is a CW-complex and that the basepoint e is a 0-cell. Show that each $J_n(X)$ and hence $J(X)$ naturally inherits the structure of a CW-complex.

(c) Show that the inclusion $\iota\colon X = J_1(X) \hookrightarrow J(X)$ satisfies the following universal property: for any map $f\colon X \to M$ to a topological monoid, there exists a unique map of topological monoids $\hat{f}\colon J(X) \to M$ such that $\hat{f} \circ \iota = f$.

(d) Conclude that the adjunction unit $X \to \Omega\Sigma X$ factors as the map ι followed by a map $g\colon J(X) \to \Omega\Sigma X$.

Remark: It is a theorem of I. M. James that the map $g\colon J(X) \to \Omega\Sigma X$ is a weak equivalence for every connected CW-complex X.

Exercise 2. (5 points)

(a) Show that for simply-connected X, the suspension homomorphism

$$\Sigma\colon \pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$$

can be identified with the map $\pi_i(\iota)$, where $\iota\colon X \to J(X)$ is as in Exercise 1. You may assume without proof the statement of the theorem given at the end of Exercise 1.

(b) Show that, for $X = S^2$ and $i = 3$, the kernel of Σ is generated by the Whitehead product $[\mathrm{id}_{S^2}, \mathrm{id}_{S^2}]$. (*Hint:* Consider the long exact sequence of the pair $(J(S^2), S^2)$.)

(c) Conclude that $\pi_4(S^3)$ is a group with two elements.

Exercise 3. (4 points) Show that every cohomology theory is represented by an Ω-spectrum.

Exercise 4. (5 points) Let (X, A) be a CW pair with both X and A connected, such that the homotopy fibre of the inclusion $A \hookrightarrow X$ is a $K(\pi, n)$, for $n \geqslant 1$. Show that $A \hookrightarrow X$ is principal if the action of $\pi_1(A)$ on $\pi_{n+1}(X, A)$ is trivial. (*Hint:* Consider the homomorphism $\pi_*(X, A) \to \pi_*(X/A)$.)