Exercise sheet 9

Due before the lecture on Monday, 7 January 2019.

For the first two exercises, consider the extension problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ Y & & \end{array} \qquad (1)$$

where we assume that (Y, A) is a CW-pair and that X is path-connected and is a simple space, or, equivalently, X admits a Postnikov tower of principal fibrations

$$\cdots \to X_3 \to X_2 \to X_1 \to X_0 = *.$$

Exercise 1. (5 points) For $n > 1$, assume that we have lifted the constant map $Y \to X_0 = *$ to a map $Y \to X_{n-1}$ extending the map $A \to X_{n-1}$. Consider the commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & X_n & \longrightarrow & PK & & \\ \big\downarrow & \nearrow & \big\downarrow & & \big\downarrow & & \\ Y & \longrightarrow & X_{n-1} & \longrightarrow & K & = & K(\pi_n(X), n+1), \end{array} \qquad (2)$$

where the right-hand square is a pullback square exhibiting X_n as the homotopy fibre of the map $X_{n-1} \to K$.

(a) Show that the map $A \to X_n$ gives rise to a nullhomotopy of the composite $A \to X_{n-1} \to K$, and, conversely, a nullhomotopy of $A \to K$ gives rise to a map $A \to X_n$ lifting the given map $A \to X_{n-1}$.

The map $Y \to X_{n-1} \to K$ together with the nullhomotopy from part (a) define a map $Y \cup_A CA \to K$ and hence a cohomology class

$$\omega_n \in H^{n+1}(Y \cup_A CA; \pi_n(X)) \cong H^{n+1}(Y, A; \pi_n(X))$$

called the *obstruction class.*

(b) Show that a lift $Y \to X_n$ for diagram (2) exists if and only ω_n vanishes.

Exercise 2. (5 points) In the situation of the extension problem (1), show that every map $A \to X$ can be extended to a map $Y \to X$ if all cohomology groups $H^{n+1}(Y, A; \pi_n(X))$ vanish.

Exercise 3. (5 points) Show without using the Dold-Thom theorem that the inclusion $S^1 = SP^1(S^1) \to SP(S^1)$ is a homotopy equivalence. (*Hint:* Identify $SP^n(\mathbb{C} \setminus \{0\}) \subseteq SP^n(\mathbb{C} \cup \{\infty\})$ with a subspace of $\mathbb{C}P^n$.)

Exercise 4. (5 points) Let h_* be a collection, indexed by $\mathbb{Z}$, of covariant functors on the category cw_* of based CW-complexes, taking values in abelian groups. Assume that h_* is equipped with natural suspension isomorphisms and that each h_n satisfies the homotopy axiom and sends cofibre sequences to exact sequences. Show that the following two statements are equivalent:

(i) For all collections $X_i \in \mathrm{cw}_*$ indexed by a countable set I, the map

$$\bigoplus_{i\in I} h_*(X_i) \longrightarrow h_*\big(\bigvee_{i\in I} X_i\big)$$

induced by the inclusions of wedge summands is an isomorphism.

(ii) If $X \in \mathrm{cw}_*$ is the colimit of a countable sequence of closed inclusions

$$X_0 \to X_1 \to X_2 \to \cdots ,$$

then the canonical map

$$\operatorname*{colim}_{n\in\mathbb{N}} h_*(X_n) \longrightarrow h_*(X)$$

induced by the maps $X_n \to X$ is an isomorphism.

(*Hint:* Identify all X_n with subspaces of X and consider the *reduced* mapping telescope obtained from

$$\bigcup_{n\geqslant 0} [n, n+1] \times X_n$$

by collapsing the subspace $\mathbb{R}_{\geqslant 0} \times \{e\}$, where $e \in X$ is the basepoint.)