Exercise sheet 10

Due before the lecture on Monday, 14 January 2019.

Exercise 1. (5 points)

(a) Construct a homeomorphism $SP^n(I) \cong \Delta^n$, where I is the unit interval and Δ^n denotes the standard topological n-simplex

$$\Delta^n = \Big\{(s_0, s_1, \dots, s_n) \in \mathbb{R}^{n+1} \;\Big|\; \forall i : s_i \geqslant 0,\ \sum_i s_i = 1\Big\}.$$

(b) Show that taking Eigenvalues defines a homeomorphism

$$U(n)/{\sim} \longrightarrow SP^n(S^1),$$

where the equivalence relation $\sim$ is conjugation by unitary matrices. (*Hint:* Show that the inverse function is continuous. To see that $U(n)/{\sim}$ is Hausdorff, show that in a Hausdorff space any two disjoint compact subsets can be separated by open sets.)

Exercise 2. (5 points) Show that $SP^2(S^1)$ is homeomorphic to a Möbius band.

Exercise 3. (5 points) Let $(p, p')\colon (E, E') \to B$ be a relative fibration over a path-connected finite-dimensional CW-complex. For a point $b \in B$, denote the relative fibre $(p^{-1}(b), p^{-1}(b) \cap E')$ by (F_b, F'_b). For a path ω in B from b to c, let (X, X') denote the pullback of (p, p') along ω and consider the *fibre transport*

$$\omega^{\#}\colon H^n(F_c, F'_c; R) \xleftarrow{\cong} H^n(X, X'; R) \xrightarrow{\cong} H^n(F_b, F'_b; R),$$

where the isomorphisms are induced by inclusions of relative fibres into (X, X'). Here, R is a fixed commutative ring and n is a fixed positive integer.

(a) Show that the assignments $b \mapsto H^n(F_b, F'_b; R)$ and $\omega \mapsto \omega^{\#}$ define a contravariant functor from the fundamental groupoid $\Pi(B)$ to the category of R-modules.

(b) Now assume that, for each $b \in B$, the R-module $H^*(F_b, F'_b; R)$ is free with a basis element in $H^n(F_b, F'_b; R)$. Show that a Thom class for the relative fibration (p, p') exists if and only if the fibre transport functor is trivial (i.e. for all $b, c \in B$, the fibre transport $\omega^{\#}$ is independent of the choice of the path ω from b to c).
(*Hint:* For the "if" part, show by induction on m that the cohomology groups $H^k(E|B^m, E'|B^m; R)$ of the restricted bundle are trivial for $k < n$ and that a Thom class for the restricted bundle exists.)

Exercise 4. (5 points) Let $(X, A) \to (Y, B)$ be a map of pairs of based spaces. Consider the following part of the long exact sequence of homotopy groups/sets:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & \pi_1(A) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(X,A) & \longrightarrow & \pi_0(A) & \longrightarrow & \pi_0(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \pi_1(B) & \longrightarrow & \pi_1(Y) & \longrightarrow & \pi_1(Y,B) & \longrightarrow & \pi_0(B) & \longrightarrow & \pi_0(Y) & \longrightarrow & \cdots
\end{array}$$

Show that the map of based sets $\pi_1(X, A) \to \pi_1(Y, B)$ is bijective if the four vertical arrows surrounding it are bijective for all choices of basepoints.
(*Hint:* Use the concatenation $\overline{\beta} * \alpha$ of the reversed path $\overline{\beta}$ with the path α as a substitute for the difference "$\alpha - \beta$".)