Exercise sheet 11

Due before the lecture on Monday, 21 January 2019.

Exercise 1. (4 points) Prove the *5-lemma modulo $\mathcal{C}$* for any Serre class $\mathcal{C}$ in the category of abelian groups. Namely, in the usual commutative diagram for the 5-lemma, the middle vertical arrow is both a $\mathcal{C}$-monomorphism and a $\mathcal{C}$-epimorphism if the four vertical arrows surrounding it are.

Exercise 2. (5 points) Let $R \subseteq \mathbb{Q}$ be a subring and let X be a simply-connected space such that $H_*(X;R) \cong H_*(S^n;R)$. Then there exists a map $S^n \to X$ that induces an isomorphism on $H_*(-;R)$.

Exercise 3. (6 points) All spaces in this exercise are assumed to be CW-complexes. A *rational homology equivalence* is a map that induces isomorphisms on $H_*(-;\mathbb{Q})$. A CW-complex Z is called *rational* if for any rational homology equivalence $f\colon X \to Y$, the induced map

$$f^*\colon [Y,Z] \longrightarrow [X,Z]$$

of (unbased) homotopy classes of (unbased) maps is a bijection. We call a map $\lambda\colon Z \to Z'$ a *rationalisation* of Z if Z' is rational and λ is a rational homology equivalence.

(a) Show that if a rationalisation $\lambda\colon Z \to Z'$ exists, it satisfies the following two universal properties:

  (i) It is "homotopy initial" among maps from Z to rational CW-complexes, i.e. for all rational CW-complexes W and all maps $f\colon Z \to W$, there is a map $f'\colon Z' \to W$, unique up to homotopy, such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} Z & \xrightarrow{f} & W \\ {\scriptstyle\lambda}\big\downarrow & \nearrow_{f'} & \\ Z' & & \end{array}$$

  (ii) It is "homotopy terminal" among rational homology equivalences whose domain is Z, i.e. for all rational homology equivalences of the form $g\colon Z \to W$, there is a map $g'\colon W \to Z'$, unique up to homotopy, such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} Z & \xrightarrow{\lambda} & Z' \\ {\scriptstyle g}\big\downarrow & \nearrow_{g'} & \\ W & & \end{array}$$

(b) Conclude that rationalisations of CW-complexes are unique up to homotopy equivalence, provided they exist.

(c) Show that a rational homology equivalence between rational CW-complexes is a homotopy equivalence.

(d) Find a rationalisation of $\mathbb{R}P^\infty$.

Exercise 4. (5 points)

(a) For which $n > 0$ is the quotient map $S^n \to \mathbb{R}P^n$ a rational homology equivalence?

(b) Construct a rational homology equivalence $\mathbb{H}P^\infty \to K(\mathbb{Z}, 4)$.