Exercise sheet 12

This exercise sheet is optional and may be handed in before the lecture on Monday, 28 January 2019. It is irrelevant for admission to the exams.

Exercise 1.

(a) Show that a vector bundle $p\colon E \to B$ of rank n is trivial if and only if there exist n sections $s_i\colon B \to E$ of p that are linearly independent at each point of B — in other words, for each $b \in B$, the vectors $s_i(b)$ for $i \in \{1, \dots, n\}$ are linearly independent elements of the vector space $p^{-1}(b)$.

(b) Deduce that the tangent bundle TS^n of the n-sphere is non-trivial when n is even.

Exercise 2. Let $\mathbb{R}^*$ denote the multiplicative group $\mathbb{R} - \{0\}$. Show that the action

$$\mathbb{R}^* \times (\mathbb{R}^2 - \{0\}) \longrightarrow \mathbb{R}^2 - \{0\}, \quad (t, (x, y)) \mapsto (tx, t^{-1}y)$$

defines a principal $\mathbb{R}^*$-bundle. Determine the orbits of this action and the orbit space (i.e. the base space of the bundle).

Exercise 3. Consider the open Möbius band

$$M = (S^1 \times \mathbb{R})/(z, -t) \sim (-z, t).$$

Show that the projection map $M \to S^1$ defines a non-trivial line bundle (vector bundle of rank 1) over S^1. (*Hint:* For non-triviality, use Exercise 1 to produce a map whose existence would contradict the intermediate value theorem of calculus.)

Exercise 4. A *smooth fibre bundle* is defined similarly to a fibre bundle, except that all spaces involved are assumed to be smooth manifolds and all maps are assumed to be smooth. Let $p\colon E \to B$ be a smooth fibre bundle and consider the tangent bundle $TE \to E$. Show that this splits as a direct sum of two vector bundles, one of which is the pullback of $TB \to B$ along the projection p. Describe the other direct summand.