**Exercise sheet 0**

These exercises are not to be handed in. They will be discussed in the tutorial sessions in week 2.

**Exercise 1.** Recall that for a pointed space $X$, there is a canonical group action of the fundamental group $\pi_1(X)$ on the homotopy groups $\pi_n(X)$ for all $n \geq 1$.

(a) For $n = 1$, show that the action is given by conjugation.

(b) Formulate an equivalent description of the action of $\pi_1(X)$ on $\pi_n(X)$ in terms of the universal cover of X, under the assumption that X admits a universal cover.

(c) For $X = \mathbb{R}P^2$, show that the above action is non-trivial for some $n$. In other words, $\mathbb{R}P^2$ is not a simple space.

**Exercise 2.** For an $H$-space $(X, x_0)$ with multiplication $\mu\colon X \times X \to X$, show that the addition on $\pi_n(X, x_0)$ can also be defined by the rule

$$[f] + [g] = [\mu \circ (f, g)]\,,$$

and that in this case, $\pi_n(X, x_0)$ is abelian also for $n = 1$.

**Exercise 3.** Let $f\colon S^n \times S^n \to S^{2n}$ be the quotient map collapsing $S^n \vee S^n$ to a point. Show that $f$ induces the zero map on all homotopy groups but $f$ is not nullhomotopic.

**Exercise 4.** Show that the set of basepoint-preserving homotopy classes $\langle X, Y\rangle$ is finite if $X$ is a finite connected CW complex and $\pi_i(Y)$ is finite for $i \leq \dim X$.

**Exercise 5.** For $m, n \geq -1$, let $X$ and $Y$ be pointed CW complexes that are $m$-connected and $n$-connected, respectively, where "$(-1)$-connected" means "non-empty". Show that the join $X * Y$ is $(m + n + 2)$-connected.
*Hint:* For higher connectivity, show that there is a homotopy equivalence $\Sigma(X \wedge Y) \simeq X * Y$ and use Hurewicz.

## Exercise sheet 1

Due before the lecture on Monday, 22 October 2018.

**Exercise 1.** (5 points) Let $X$ be a pointed space. Show that $\Sigma X$ is a co-$H$-group and that $\Sigma(\Sigma X)$ is a homotopy commutative co-$H$-group. Dualise your argument to show that $\Omega X$ is an $H$-group and that $\Omega(\Omega X)$ is a homotopy commutative $H$-group.

**Exercise 2.** (5 points) Weak equivalence is not a symmetric relation:

(a) Give an example of a weak equivalence $X \to Y$ for which there does not exist a weak equivalence $Y \to X$.

However, there is an equivalence relation $\simeq_w$ generated by weak equivalence: $X \simeq_w Y$ if there are spaces $X = X_1, X_2, \ldots, X_n = Y$ with weak equivalences $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$ for each $i$.

(b) Show that $X \simeq_w Y$ if and only if $X$ and $Y$ have a common CW approximation.

**Exercise 3.** (4 points) For $n \geq 0$, show that an $n$-connected, $n$-dimensional CW complex is contractible.

**Exercise 4.** (6 points)

(a) Show that the suspension of an acyclic CW complex (i.e. a CW complex whose reduced homology vanishes) is contractible.

(b) Show that a map between simply-connected CW complexes is a homotopy equivalence if its mapping cone is contractible.

(c) Provide an explicit counter-example to show that the connectivity assumption in (b) cannot be dropped.

**Exercise sheet 2**

Due before the lecture on Monday, 29 October 2018.

**Exercise 1.** (4 points) Let $f\colon X \to Y$ be a weak equivalence and let $A$ be a based CW-complex. Show that $f$ induces bijections $[A, X] \to [A, Y]$ and $\langle A, X\rangle \to \langle A, Y\rangle$. *Hint:* use the Compression Lemma.

**Exercise 2.** (6 points) Find a map of CW-complexes that induces isomorphisms on integral homology and on $\pi_1$, but is not a homotopy equivalence. *Hint:* For $n \geqslant 2$, recall that $\pi_n(S^1 \vee S^n) \cong \mathbb{Z}\left[t, t^{-1}\right]$ by looking at the universal cover. Construct a CW-complex $X$ from $S^1 \vee S^n$ by attaching a single $(n+1)$-cell along a map representing $2t - 1$ and consider the inclusion of the 1-skeleton into $X$.

**Exercise 3.** (5 points) Let $f\colon A \to X$ be a cofibration. Show:

(a) $f$ is an embedding.

(b) $f$ is a closed map if $X$ is Hausdorff.

**Exercise 4.** (5 points)

(a) Show that $E \to B$ is a Serre fibration if its restriction $X \to B$ is a Serre fibration for each path-component $X$ of $E$.

(b) Consider the subspace $E \subseteq \mathbb{R}^2$ given by

$$E := \big([0,1] \times \{\tfrac{1}{n} \,|\, n \in \mathbb{N}\}\big) \cup \{(x, x-1) \,|\, 0 \leqslant x \leqslant 1\}.$$

Show that the map

$$p\colon E \to [0,1]\,,\ (x,y) \mapsto x$$

is a Serre fibration, but not a Hurewicz fibration, i.e. that $p$ satisfies the homotopy lifting property with respect to all CW-complexes, but not with respect to arbitrary spaces.
(*Remark:* One can show that a Serre fibration between CW-complexes is always a Hurewicz fibration.)

**Exercise sheet 3**

Due before the lecture on Monday, 5 November 2018.

**Exercise 1.** (5 points) Let $X$ be a topological space and write

$$\Delta = \{(x,x) \mid x \in X\} \subseteq X \times X$$

for the diagonal.

(a) Show that $X$ is Hausdorff if and only if $\Delta \subseteq X \times X$ is closed with respect to the product topology.

(b) Assume that $X$ is compactly generated. Show that $X$ is weakly Hausdorff if and only if $\Delta \subseteq X \times X$ is closed with respect to the compactly generated product topology, i.e. the $k$-ification of the usual product topology.

**Exercise 2.** (4 points) Consider the trivial fibre bundle $I \times I \to I$. Find a subspace $E \subseteq I \times I$ such that the projection restricts to a fibration $E \to I$ that is not a fibre bundle.

**Exercise 3.** (5 points)

(a) Show that $\pi_n(S^n) \cong \mathbb{Z}$.

(b) Assume that $S^m \to S^n$ is a fibre bundle with fibre $S^k$. Show that $k = n-1$ and $m = 2n-1$.

(c) Let $E \to B$ be a fibration over a path-connected space $B$ such that the inclusion of the fibre $F \to E$ is null-homotopic. Construct isomorphisms $\pi_n(B) \cong \pi_n(E) \times \pi_{n-1}(F)$. Deduce that $\pi_7(S^4)$ and $\pi_{15}(S^8)$ contain $\mathbb{Z}$ as a summand. (You may use without proof the fact that there exist fibre bundles as in (b) for $n = 4$ and $n = 8$.)

**Exercise 4.** (6 points) Let $A \to X$ be a cofibration of spaces and let $f$ and $g$ be homotopic maps $X \to Y$ that agree on $A$.

(a) Show that if $A$ is contractible, then $f$ and $g$ are homotopic relative to $A$. (*Hint:* Fix a homotopy $h\colon f \simeq g$. Choose a null-homotopy $k$ of $f\big|_A$ and use it to construct a homotopy of homotopies $K\colon A \times I \times I \to Y$ between $h\big|_A$ and the constant homotopy $f\big|_A \simeq g\big|_A$. Then apply the homotopy extension property of $A \times I \to X \times I$.)

(b) Show that the contractibility assumption in (a) is necessary. (*Hint:* Consider maps of pairs $(D^1, S^0) \to (S^1, *)$.)

## Exercise sheet 3

## Correction to exercise 4a

As written, Ex. 4a of sheet 3 is false (thanks to Sil Linskens and Tim Santens for a counterexample!). However, it is correct with stronger hypotheses. The corrected version of it is as follows:

**Exercise 4(a).** Let $A \to X$ be a cofibration of spaces and let $f, g \colon X \to Y$ be continuous maps such that $f \simeq g$ and $f|_A = g|_A$.
Assume that $Y$ is simply-connected, $A$ is contractible and that the inclusion $\{a\} \hookrightarrow A$ is a closed cofibration for some $a \in A$.

(a)$_1$ Show that any map $A \times S^1 \to Y$ extends to $A \times D^2 \to Y$.

(a)$_2$ Using the homotopy extension property of $A \times [0,1] \to X \times [0,1]$, show that $f$ and $g$ are homotopic relative to $A$.

**Exercise sheet 4**

Due before the lecture on Monday, 12 November 2018.

**Exercise 1.** (5 points) Let $n \geqslant 2$ and let $X$ be a homology $n$-sphere, i.e. a space such that

$$\widetilde{H}_i(X;\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z} & i = n \\ 0 & i \neq n. \end{cases}$$

Show: if $X$ is a simply-connected CW complex, then it is homotopy equivalent to $S^n$.

**Exercise 2.** (5 points) Compute all of the homotopy groups of $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$. (*Hint:* consider suitable fibre bundles $S^\infty \cong \operatorname{colim}_n S^n \to \mathbb{R}P^\infty$ respectively $S^\infty \cong \operatorname{colim}_n S^{2n+1} \to \mathbb{C}P^\infty$ and use Exercise 4b.)

**Exercise 3.** (4 points) Consider the trivial fibre bundle $I \times I \to I$. Find a subspace $E \subseteq I \times I$ such that the projection restricts to a quasi-fibration $E \to I$ that is not a Serre fibration.

**Exercise 4.** (6 points) Let $X_0 \to X_1 \to X_2 \to \cdots$ be a countably infinite sequence of closed inclusions of $T_1$ spaces and write $X = \operatorname{colim}_i X_i$ for the colimit of this sequence. (Recall that a space is $T_1$ if points are closed.)

(a) Show that the canonical map

$$\operatorname*{colim}_i \pi_n(X_i) \longrightarrow \pi_n(X)$$

induced by the inclusions $X_i \to X$ is a bijection for all $n$. (*Hint:* show that if $K$ is compact, then the image of any map $K \to X$ must be contained in some $X_i$.)

(b) Deduce that $S^\infty \cong \operatorname{colim}_i S^i$ is contractible.

(c) Assume that in the following commutative diagram of $T_1$ spaces, all horizontal maps are closed inclusions and all vertical maps are weak equivalences.

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots \end{array}$$

Show that the induced map

$$f\colon X = \operatorname*{colim}_i X_i \longrightarrow \operatorname*{colim}_i Y_i = Y$$

is a weak equivalence.

*Remark:* One can show that weak Hausdorff spaces are $T_1$ and that sequential colimits along closed inclusions in the category of compactly generated weak Hausdorff spaces can be computed in the category of all topological spaces. Consequently, the above statements also hold in the "convenient category of spaces".

## Exercise sheet 5

Due before the lecture on Monday, 19 November 2018.

**Exercise 1.** (Left properness) (5 points) For a pushout

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ B & \longrightarrow & D \end{array}$$

in the category CGWH show the following:

(a) If $i$ is a cofibration, then so is $i'$.

(b) If $i$ is a cofibration and a homotopy equivalence, then so is $i'$.

You may use without proof the fact that pushouts in CGWH with one “leg” a cofibration can be computed in Top.
(*Hint:* For (b), first show that $i$ is the inclusion of a deformation retract, i.e. there exists $r\colon B \to A$ such that $r \circ i = \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_B$ rel. $A$.)

**Exercise 2.** (5 points) Let $\eta\colon S^3 \to S^2$ denote the Hopf fibration and consider the sequence of suspension homomorphisms

$$\pi_1(S^0) \xrightarrow{S} \pi_2(S^1) \xrightarrow{S} \pi_3(S^2) \xrightarrow{S} \pi_4(S^3) \xrightarrow{S} \pi_5(S^4) \xrightarrow{S} \cdots$$

(a) Show that the sequence stabilizes at $\pi_4(S^3)$ and that the latter group is generated by $S([\eta])$.

(b) Show that $2 \cdot S([\eta]) = 0$ in $\pi_4(S^3)$. Deduce that this group is either trivial or has two elements. (*Hint:* First show that $\eta \circ c = r \circ \eta$ where $c$ is complex conjugation in both coordinates of unit vectors in $\mathbb{C}^2$ and $r$ is a suitable reflection of $S^2$.)

*Remark:* Indeed $\pi_4(S^3) \cong \mathbb{Z}/2$, but we don't have the technology available to show that $S([\eta]) \neq 0$ yet. This is usually done via cohomology operations.

**Exercise 3.** (5 points) Let $\mathbb{Q}$ be the rational numbers equipped with the subspace topology of $\mathbb{R}$. Topologise $X = \mathbb{Q} \cup \{\infty\}$ such that $U \subseteq X$ is open if and only if either $\infty \notin U$ and $U$ is open in $\mathbb{Q}$, or $\infty \in U$ and $X - U$ is compact. Show that $X$ is a weak Hausdorff space that is not Hausdorff.

**Exercise 4.** (5 points) For maps of based spaces $f\colon S^i \to X$ and $g\colon S^j \to X$, define their *Whitehead product* $[f, g]$ to be the composition

$$[f, g] : S^{i+j-1} \longrightarrow S^i \vee S^j \xrightarrow{f \vee g} X,$$

where the first map is the attaching map of the top-dimensional cell of $S^i \times S^j$ with its standard cell structure.

(a) Show that this construction gives rise to a well-defined operation

$$[-\,,-] : \pi_i(X) \times \pi_j(X) \longrightarrow \pi_{i+j-1}(X),$$

which is bilinear when $i, j \geqslant 2$.

(b) Identify the Whitehead product for $i = j = 1$ and justify the square bracket notation.

## Exercise sheet 6

Due before the lecture on Monday, 26 November 2018.

**Exercise 1.** (5 points) Recall that we introduced the *Whitehead product* on the previous exercise sheet.

(a) Show that $[\mathrm{id}_{S^2}, \mathrm{id}_{S^2}] = 2 \cdot [\eta]$ in $\pi_3(S^2)$, where $\eta \colon S^3 \to S^2$ is the Hopf map.

(*Hint:* Consider the cohomology ring of the homotopy cofibre of a map representing $[\mathrm{id}_{S^2}, \mathrm{id}_{S^2}]$ and use the fact that the Hopf invariant (introduced in Topology II, exercise 8.2) defines an isomorphism $\pi_3(S^2) \cong \mathbb{Z}$.)

(b) Show that all Whitehead products of classes $\alpha \in \pi_i(X)$, $\beta \in \pi_j(X)$ lie in the kernel of the suspension homomorphism

$$\Sigma \colon \pi_{i+j-1}(X) \longrightarrow \pi_{i+j}(\Sigma X).$$

(*Hint:* First show that the suspension of the inclusion $S^i \vee S^j \hookrightarrow S^i \times S^j$ admits a retraction up to homotopy.)

Together, (a) and (b) give an alternative proof of the fact that the group $\pi_4(S^3)$ has at most two elements.

**Exercise 2.** (4 points) Show that if $X$ is a path-connected H-space, then all Whitehead products on $\pi_*(X)$ vanish.

**Exercise 3.** (6 points) For $p, q \geqslant 1$, let $X$ and $Y$ be well-based spaces that are $p$- and $q$-connected, respectively.

(a) Show that for $i \geqslant 2$, the long exact sequence of $(X \times Y, X \vee Y)$ gives rise to a split short exact sequence

$$0 \to \pi_{i+1}(X \times Y, X \vee Y) \to \pi_i(X \vee Y) \to \pi_i(X \times Y) \to 0.$$

(b) Show that the composite $\pi_i(X) \to \pi_i(X \vee Y) \to \pi_i(X \vee Y, Y)$ is an isomorphism for $i \leqslant p + q$ (and similarly for $X$ and $Y$ switched). (*Hint:* Compose with the projection to show injectivity and use the Blakers-Massey theorem for surjectivity.)

(c) Show that the inclusions of wedge summands induce an isomorphism

$$\pi_i(X) \times \pi_i(Y) \longrightarrow \pi_i(X \vee Y)$$

for $i \leqslant p+q$ and conclude from (a) that $\pi_i(X \times Y, X \vee Y) = 0$ for $i \leqslant p+q+1$.

(d) Compute $\pi_n(S^n \vee S^n)$ for $n \geqslant 2$.

**Exercise 4.** (5 points) Let $X$ be a based CW-complex. Show that the contravariant functor $\langle - , X \rangle$ from based CW-complexes to sets is *half-exact*, i.e. it is homotopy invariant and satisfies the wedge and Mayer-Vietoris axioms. This is therefore a necessary condition for Brown's representability theorem.

**Exercise sheet 7**

Due before the lecture on Monday, 3 December 2018.

For the first two exercises, call $X$ an *Eilenberg-MacLane space of type* $K(G,n)$ if $\pi_n(X) \cong G$ and all other homotopy groups of $X$ vanish. This is equivalent to the definition (via the representability theorem of E. Brown) given in lectures, as can be seen by combining Exercise 2 with the calculation of homotopy groups discussed in the lectures.

**Exercise 1.** (5 points) Let $n \geqslant 2$ and let $G$ be an abelian group with presentation

$$0 \to \bigoplus_{j\in J} \mathbb{Z} \xrightarrow{\varphi} \bigoplus_{i\in I} \mathbb{Z} \longrightarrow G \to 0.$$

(a) Choose a map

$$f\colon \bigvee_{j\in J} S^n \longrightarrow \bigvee_{i\in I} S^n$$

so that $H_n(f;\mathbb{Z}) = \varphi$ under a chosen identification $H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$. Show that the homotopy cofibre $X = C_f$ of $f$ is $(n-1)$-connected and satisfies $\pi_n(X) \cong G$.

(b) Construct an Eilenberg-MacLane space of type $K(G,n)$ by attaching cells of dimension at least $n+2$ to $X$.

**Exercise 2.** (5 points) Assume that $K$ and $K'$ are CW-complexes that are both Eilenberg-MacLane spaces of type $K(G,n)$, $n \geqslant 2$. Show that $K$ and $K'$ are homotopy equivalent. (*Hint:* If $K$ is constructed as in Exercise 1, find a map $g\colon X \to K'$ that induces an isomorphism on $\pi_n$ and show that $g$ extends to all of $K$.)

**Exercise 3.** (5 points) Let $X = \mathrm{colim}_n(X^n)$ be a based CW-complex, written as the colimit of its skeleta $X^0 \hookrightarrow X^1 \hookrightarrow X^2 \hookrightarrow X^3 \hookrightarrow \cdots$, and let $Y$ be any based space. Consider the sequence of based sets

$$* \to Ph(X,Y) \longrightarrow \langle X,Y\rangle \longrightarrow \lim_n \langle X^n, Y\rangle \to *, \tag{1}$$

where $Ph(X,Y) \subseteq \langle X,Y\rangle$ is the subset consisting of based homotopy classes of phantom maps and the second map is induced by the inclusions $i_n\colon X^n \hookrightarrow X$. Show that (1) is an exact sequence of based sets, and show that it is an exact sequence of groups if $X$ is a suspension.

**Exercise 4.** (5 points) Let $\theta\colon T \to U$ be a natural transformation between half-exact contravariant functors defined on the category $\mathrm{CW}_*$ of based, connected CW-complexes and taking values in the category of abelian groups.[1]

(a) For a fixed $n > 0$, assume that the function $\theta(S^m)\colon T(S^m) \to U(S^m)$ is bijective for all $m < n$ and surjective for $m = n$. Show that, for every based, connected, finite-dimensional CW-complex $X$, the function

$$\theta(X)\colon T(X) \longrightarrow U(X) \tag{2}$$

is bijective for $\dim(X) < n$ and surjective for $\dim(X) = n$.

(b) Now assume that the function $\theta(S^m)\colon T(S^m) \to U(S^m)$ is bijective for all $m > 0$. Show that the function (2) is bijective for every based, connected CW-complex $X$.

[1] The following statements are also true more generally when $T$ and $U$ take values in the category of sets, but the proof of part (a) in this more general setting is much more involved.

**Exercise sheet 8**

Due before the lecture on Monday, 10 December 2018.

**Exercise 1.** (6 points) For a based space $(Y, y_0)$, let $\Omega Y$ denote its based loop space equipped with the usual H-space structure given by concatenation of based loops. Define the *Moore loop space* of $Y$ to be the space

$$\Omega' Y = (\{0\} \times \{*\}) \cup (\mathbb{R}_{>0} \times \Omega Y),$$

topologised as a subspace of $\mathbb{R}_{\geqslant 0} \times \Omega Y$, where $*$ is the constant loop at the basepoint. It is a (strictly associative and strictly unital) topological monoid with multiplication $\Omega' Y \times \Omega' Y \to \Omega' Y$ given by

$$(t_1, \gamma_1) \cdot (t_2, \gamma_2) = (t_1 + t_2, \gamma)$$

where $\gamma$ is the based loop

$$\gamma(s) = \begin{cases} \gamma_1\left(\frac{s(t_1+t_2)}{t_1}\right) & 0 \leqslant s \leqslant \frac{t_1}{t_1+t_2} \\ \gamma_2\left(\frac{s(t_1+t_2)-t_1}{t_2}\right) & \frac{t_1}{t_1+t_2} \leqslant s \leqslant 1. \end{cases}$$

(These statements may be assumed without proof.)

(a) Show that the map

$$\lambda \colon \Omega Y \longrightarrow \Omega' Y, \;\; \gamma \longmapsto (1, \gamma)$$

is a homotopy equivalence. Is it a map of H-spaces?

Now let $(X, e)$ be a based space. For $n \geqslant 2$, the $n$-fold *reduced product* is the quotient space

$$J_n(X) = X^n / \sim$$

by the equivalence relation generated by

$$(x_1, \ldots, x_i, e, \ldots, x_n) \sim (x_1, \ldots, e, x_i, \ldots, x_n)$$

for all $1 \leqslant i \leqslant n-1$. We set $J_1(X) = X$. For each $n \geqslant 1$, there is an inclusion $J_n(X) \hookrightarrow J_{n+1}(X)$ given by inserting a basepoint in one of the coordinates. We call

$$J(X) = \operatorname*{colim}_n J_n(X)$$

the *James construction* on $X$.

(b) Assume that $X$ is a CW-complex and that the basepoint $e$ is a 0-cell. Show that each $J_n(X)$ and hence $J(X)$ naturally inherits the structure of a CW-complex.

(c) Show that the inclusion $\iota\colon X = J_1(X) \hookrightarrow J(X)$ satisfies the following universal property: for any map $f\colon X \to M$ to a topological monoid, there exists a unique map of topological monoids $\hat{f}\colon J(X) \to M$ such that $\hat{f} \circ \iota = f$.

(d) Conclude that the adjunction unit $X \to \Omega\Sigma X$ factors as the map $\iota$ followed by a map $g\colon J(X) \to \Omega\Sigma X$.

*Remark:* It is a theorem of I. M. James that the map $g\colon J(X) \to \Omega\Sigma X$ is a weak equivalence for every connected CW-complex $X$.

**Exercise 2.** (5 points)

(a) Show that for simply-connected $X$, the suspension homomorphism

$$\Sigma\colon \pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$$

can be identified with the map $\pi_i(\iota)$, where $\iota\colon X \to J(X)$ is as in Exercise 1. You may assume without proof the statement of the theorem given at the end of Exercise 1.

(b) Show that, for $X = S^2$ and $i = 3$, the kernel of $\Sigma$ is generated by the Whitehead product $[\mathrm{id}_{S^2}, \mathrm{id}_{S^2}]$. (*Hint:* Consider the long exact sequence of the pair $(J(S^2), S^2)$.)

(c) Conclude that $\pi_4(S^3)$ is a group with two elements.

**Exercise 3.** (4 points) Show that every cohomology theory is represented by an $\Omega$-spectrum.

**Exercise 4.** (5 points) Let $(X, A)$ be a CW pair with both $X$ and $A$ connected, such that the homotopy fibre of the inclusion $A \hookrightarrow X$ is a $K(\pi, n)$, for $n \geqslant 1$. Show that $A \hookrightarrow X$ is principal if the action of $\pi_1(A)$ on $\pi_{n+1}(X, A)$ is trivial. (*Hint:* Consider the homomorphism $\pi_*(X, A) \to \pi_*(X/A)$.)

**Exercise sheet 9**

Due before the lecture on Monday, 7 January 2019.

For the first two exercises, consider the extension problem

$$\begin{array}{ccc} A & \longrightarrow & X \\ \big\downarrow & \nearrow & \\ Y & & \end{array} \qquad (1)$$

where we assume that $(Y, A)$ is a CW-pair and that $X$ is path-connected and is a simple space, or, equivalently, $X$ admits a Postnikov tower of principal fibrations

$$\cdots \to X_3 \to X_2 \to X_1 \to X_0 = *.$$

**Exercise 1.** (5 points) For $n > 1$, assume that we have lifted the constant map $Y \to X_0 = *$ to a map $Y \to X_{n-1}$ extending the map $A \to X_{n-1}$. Consider the commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & X_n & \longrightarrow & PK & & \\ \big\downarrow & \nearrow & \big\downarrow & & \big\downarrow & & \\ Y & \longrightarrow & X_{n-1} & \longrightarrow & K & = & K(\pi_n(X), n+1), \end{array} \qquad (2)$$

where the right-hand square is a pullback square exhibiting $X_n$ as the homotopy fibre of the map $X_{n-1} \to K$.

(a) Show that the map $A \to X_n$ gives rise to a nullhomotopy of the composite $A \to X_{n-1} \to K$, and, conversely, a nullhomotopy of $A \to K$ gives rise to a map $A \to X_n$ lifting the given map $A \to X_{n-1}$.

The map $Y \to X_{n-1} \to K$ together with the nullhomotopy from part (a) define a map $Y \cup_A CA \to K$ and hence a cohomology class

$$\omega_n \in H^{n+1}(Y \cup_A CA; \pi_n(X)) \cong H^{n+1}(Y, A; \pi_n(X))$$

called the *obstruction class.*

(b) Show that a lift $Y \to X_n$ for diagram (2) exists if and only $\omega_n$ vanishes.

**Exercise 2.** (5 points) In the situation of the extension problem (1), show that every map $A \to X$ can be extended to a map $Y \to X$ if all cohomology groups $H^{n+1}(Y, A; \pi_n(X))$ vanish.

**Exercise 3.** (5 points) Show without using the Dold-Thom theorem that the inclusion $S^1 = SP^1(S^1) \to SP(S^1)$ is a homotopy equivalence. (*Hint:* Identify $SP^n(\mathbb{C} \setminus \{0\}) \subseteq SP^n(\mathbb{C} \cup \{\infty\})$ with a subspace of $\mathbb{C}P^n$.)

**Exercise 4.** (5 points) Let $h_*$ be a collection, indexed by $\mathbb{Z}$, of covariant functors on the category $\mathrm{cw}_*$ of based CW-complexes, taking values in abelian groups. Assume that $h_*$ is equipped with natural suspension isomorphisms and that each $h_n$ satisfies the homotopy axiom and sends cofibre sequences to exact sequences. Show that the following two statements are equivalent:

(i) For all collections $X_i \in \mathrm{cw}_*$ indexed by a countable set $I$, the map

$$\bigoplus_{i\in I} h_*(X_i) \longrightarrow h_*\big(\bigvee_{i\in I} X_i\big)$$

induced by the inclusions of wedge summands is an isomorphism.

(ii) If $X \in \mathrm{cw}_*$ is the colimit of a countable sequence of closed inclusions

$$X_0 \to X_1 \to X_2 \to \cdots ,$$

then the canonical map

$$\operatorname*{colim}_{n\in\mathbb{N}} h_*(X_n) \longrightarrow h_*(X)$$

induced by the maps $X_n \to X$ is an isomorphism.

(*Hint:* Identify all $X_n$ with subspaces of $X$ and consider the *reduced* mapping telescope obtained from

$$\bigcup_{n\geqslant 0} [n, n+1] \times X_n$$

by collapsing the subspace $\mathbb{R}_{\geqslant 0} \times \{e\}$, where $e \in X$ is the basepoint.)

## Exercise sheet 10

Due before the lecture on Monday, 14 January 2019.

**Exercise 1.** (5 points)

(a) Construct a homeomorphism $SP^n(I) \cong \Delta^n$, where $I$ is the unit interval and $\Delta^n$ denotes the standard topological $n$-simplex

$$\Delta^n = \Big\{(s_0, s_1, \dots, s_n) \in \mathbb{R}^{n+1} \;\Big|\; \forall i : s_i \geqslant 0,\ \sum_i s_i = 1\Big\}.$$

(b) Show that taking Eigenvalues defines a homeomorphism

$$U(n)/{\sim} \longrightarrow SP^n(S^1),$$

where the equivalence relation $\sim$ is conjugation by unitary matrices. (*Hint:* Show that the inverse function is continuous. To see that $U(n)/{\sim}$ is Hausdorff, show that in a Hausdorff space any two disjoint compact subsets can be separated by open sets.)

**Exercise 2.** (5 points) Show that $SP^2(S^1)$ is homeomorphic to a Möbius band.

**Exercise 3.** (5 points) Let $(p, p')\colon (E, E') \to B$ be a relative fibration over a path-connected finite-dimensional CW-complex. For a point $b \in B$, denote the relative fibre $(p^{-1}(b), p^{-1}(b) \cap E')$ by $(F_b, F_b')$. For a path $\omega$ in $B$ from $b$ to $c$, let $(X, X')$ denote the pullback of $(p, p')$ along $\omega$ and consider the *fibre transport*

$$\omega^{\#}\colon H^n(F_c, F_c'; R) \xleftarrow{\cong} H^n(X, X'; R) \xrightarrow{\cong} H^n(F_b, F_b'; R),$$

where the isomorphisms are induced by inclusions of relative fibres into $(X, X')$. Here, $R$ is a fixed commutative ring and $n$ is a fixed positive integer.

(a) Show that the assignments $b \mapsto H^n(F_b, F_b'; R)$ and $\omega \mapsto \omega^{\#}$ define a contravariant functor from the fundamental groupoid $\Pi(B)$ to the category of $R$-modules.

(b) Now assume that, for each $b \in B$, the $R$-module $H^*(F_b, F_b'; R)$ is free with a basis element in $H^n(F_b, F_b'; R)$. Show that a Thom class for the relative fibration $(p, p')$ exists if and only if the fibre transport functor is trivial (i.e. for all $b, c \in B$, the fibre transport $\omega^{\#}$ is independent of the choice of the path $\omega$ from $b$ to $c$).
(*Hint:* For the "if" part, show by induction on $m$ that the cohomology groups $H^k(E|B^m, E'|B^m; R)$ of the restricted bundle are trivial for $k < n$ and that a Thom class for the restricted bundle exists.)

**Exercise 4.** (5 points) Let $(X, A) \to (Y, B)$ be a map of pairs of based spaces. Consider the following part of the long exact sequence of homotopy groups/sets:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & \pi_1(A) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(X,A) & \longrightarrow & \pi_0(A) & \longrightarrow & \pi_0(X) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \pi_1(B) & \longrightarrow & \pi_1(Y) & \longrightarrow & \pi_1(Y,B) & \longrightarrow & \pi_0(B) & \longrightarrow & \pi_0(Y) & \longrightarrow & \cdots
\end{array}$$

Show that the map of based sets $\pi_1(X, A) \to \pi_1(Y, B)$ is bijective if the four vertical arrows surrounding it are bijective for all choices of basepoints.
(*Hint:* Use the concatenation $\overline{\beta} * \alpha$ of the reversed path $\overline{\beta}$ with the path $\alpha$ as a substitute for the difference "$\alpha - \beta$".)

**Exercise sheet 11**

Due before the lecture on Monday, 21 January 2019.

**Exercise 1.** (4 points) Prove the *5-lemma modulo $\mathcal{C}$* for any Serre class $\mathcal{C}$ in the category of abelian groups. Namely, in the usual commutative diagram for the 5-lemma, the middle vertical arrow is both a $\mathcal{C}$-monomorphism and a $\mathcal{C}$-epimorphism if the four vertical arrows surrounding it are.

**Exercise 2.** (5 points) Let $R \subseteq \mathbb{Q}$ be a subring and let $X$ be a simply-connected space such that $H_*(X; R) \cong H_*(S^n; R)$. Then there exists a map $S^n \to X$ that induces an isomorphism on $H_*(-; R)$.

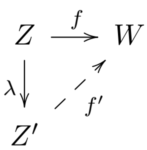
**Exercise 3.** (6 points) All spaces in this exercise are assumed to be CW-complexes. A *rational homology equivalence* is a map that induces isomorphisms on $H_*(-; \mathbb{Q})$. A CW-complex $Z$ is called *rational* if for any rational homology equivalence $f\colon X \to Y$, the induced map

$$f^*\colon [Y, Z] \longrightarrow [X, Z]$$

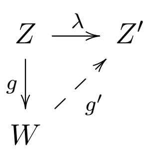
of (unbased) homotopy classes of (unbased) maps is a bijection. We call a map $\lambda\colon Z \to Z'$ a *rationalisation* of $Z$ if $Z'$ is rational and $\lambda$ is a rational homology equivalence.

(a) Show that if a rationalisation $\lambda\colon Z \to Z'$ exists, it satisfies the following two universal properties:

(i) It is "homotopy initial" among maps from $Z$ to rational CW-complexes, i.e. for all rational CW-complexes $W$ and all maps $f\colon Z \to W$, there is a map $f'\colon Z' \to W$, unique up to homotopy, such that the following diagram commutes up to homotopy:




(ii) It is "homotopy terminal" among rational homology equivalences whose domain is $Z$, i.e. for all rational homology equivalences of the form $g\colon Z \to W$, there is a map $g'\colon W \to Z'$, unique up to homotopy, such that the following diagram commutes up to homotopy:



(b) Conclude that rationalisations of CW-complexes are unique up to homotopy equivalence, provided they exist.

(c) Show that a rational homology equivalence between rational CW-complexes is a homotopy equivalence.

(d) Find a rationalisation of $\mathbb{R}P^\infty$.

**Exercise 4.** (5 points)

(a) For which $n > 0$ is the quotient map $S^n \to \mathbb{R}P^n$ a rational homology equivalence?

(b) Construct a rational homology equivalence $\mathbb{H}P^\infty \to K(\mathbb{Z}, 4)$.

## Exercise sheet 12

This exercise sheet is optional and may be handed in before the lecture on Monday, 28 January 2019. It is irrelevant for admission to the exams.

**Exercise 1.**

(a) Show that a vector bundle $p\colon E \to B$ of rank $n$ is trivial if and only if there exist $n$ sections $s_i\colon B \to E$ of $p$ that are linearly independent at each point of $B$ — in other words, for each $b \in B$, the vectors $s_i(b)$ for $i \in \{1, \dots, n\}$ are linearly independent elements of the vector space $p^{-1}(b)$.

(b) Deduce that the tangent bundle $TS^n$ of the $n$-sphere is non-trivial when $n$ is even.

**Exercise 2.** Let $\mathbb{R}^*$ denote the multiplicative group $\mathbb{R} - \{0\}$. Show that the action

$$\mathbb{R}^* \times (\mathbb{R}^2 - \{0\}) \longrightarrow \mathbb{R}^2 - \{0\}, \quad (t, (x, y)) \mapsto (tx, t^{-1}y)$$

defines a principal $\mathbb{R}^*$-bundle. Determine the orbits of this action and the orbit space (i.e. the base space of the bundle).

**Exercise 3.** Consider the open Möbius band

$$M = (S^1 \times \mathbb{R})/(z, -t) \sim (-z, t).$$

Show that the projection map $M \to S^1$ defines a non-trivial line bundle (vector bundle of rank 1) over $S^1$. (*Hint:* For non-triviality, use Exercise 1 to produce a map whose existence would contradict the intermediate value theorem of calculus.)

**Exercise 4.** A *smooth fibre bundle* is defined similarly to a fibre bundle, except that all spaces involved are assumed to be smooth manifolds and all maps are assumed to be smooth. Let $p\colon E \to B$ be a smooth fibre bundle and consider the tangent bundle $TE \to E$. Show that this splits as a direct sum of two vector bundles, one of which is the pullback of $TB \to B$ along the projection $p$. Describe the other direct summand.

## Revision sheet

The following exercises cover some of the topics of the lecture and will be discussed in the exercise sessions on 1 February.

**Exercise 1.** Decide whether the following statements are true or false.

(1) The suspension homomorphism $\Sigma\colon \pi_3(S^2) \to \pi_4(S^3)$ is injective.

(2) If $p\colon E \to B$ is a Serre fibration over a path-connected space $B$, then all fibres $p^{-1}(b)$ for $b \in B$ are weakly homotopy equivalent.

(3) For all CGWH spaces $X, Y$ and $Z$, there is a natural homeomorphism

$$\mathrm{Map}(X \times Y, Z) \longrightarrow \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

(4) Every basepoint-preserving map $\mathbb{R}P^\infty \to \mathbb{C}P^\infty$ is based homotopic to a constant map.

(5) For any based CW-complex $Z$, the functor $\langle -, Z\rangle$ from the category of based CW-complexes to the category of pointed sets is half-exact.

(6) Let $h_*$ and $k_*$ be reduced homology theories defined on the category of based CW-complexes such that $h_i(S^0) \cong k_i(S^0)$ for all $i \in \mathbb{Z}$. Then any isomorphism of groups $h_0(S^0) \to k_0(S^0)$ extends to an isomorphism of homology theories $h_* \cong k_*$.

(7) Let $X$ and $Y$ be CW-complexes. Then there exists a weak equivalence $SP(X \vee Y) \to SP(X) \times SP(Y)$.

(8) Let $p$ be a prime number. Then the class of all finitely generated abelian groups $A$ such that $a \mapsto p \cdot a\colon A \to A$ is invertible forms a Serre class.

(9) Let $G$ be a discrete group and $E$ a free right $G$-space. Then the orbit map $E \to E/G$ is a principal $G$-bundle.

(10) There are infinitely many pairwise non-isomorphic principal $\mathbb{Z}$-bundles over the base space $S^1$.

**Exercise 2.** Assume that the following diagram is a pushout diagram in the category of topological spaces.

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow[g]{} & D \end{array}$$

Prove that, if $i$ is a cofibration, then $j$ is a cofibration. Is this statement true if we work instead in the category CGWH?

**Exercise 3.** Let $X$ be a path-connected space and let $\Sigma X$ denote its reduced suspension. Show that, for $n \geqslant 3$, there are isomorphisms

$$\pi_n(\Sigma X, X) \cong \pi_n(\Sigma X) \times \pi_{n-1}(X).$$

**Exercise 4.** Show that for any two abelian groups $A, B$ and any $n \geqslant 2$ there is a bijection

$$\mathrm{Hom}(A, B) \cong \langle K(A, n), K(B, n) \rangle$$

between the set of group homomorphisms $A \to B$ and the set of based homotopy classes of based maps $K(A, n) \to K(B, n)$ between the corresponding Eilenberg-MacLane spaces.

**Exercise 5.** Compute $\pi_*(\mathbb{C}P^n) \otimes \mathbb{Q}$.

**Exercise 6.** Find maps $f_i \colon X_i \to Y_i$ for $i = 1, 2$ between simply-connected CW-complexes such that:

(a) $f_1$ induces a surjection on $\pi_*$ but not on $H_*$,

(b) $f_2$ induces a surjection on $H_*$ but not on $\pi_*$.

(*Hint:* Only consider spheres and projective spaces. For (b), use Exercise 5 to deduce that $\pi_7(\mathbb{C}P^2)$ is finite.)

**Extra exercises (optional!)**

Some optional extra exercises, not for handing in.

**Exercise A.** Let $X$ and $Y$ be compact, connected manifolds with empty boundary, with $X$ orientable and $Y$ non-orientable, such that $\pi_1(X) \cong \pi_1(Y)$ and this group is finite. Use these to construct two non-homotopy-equivalent spaces with isomorphic homotopy groups. Why does this not contradict Whitehead's theorem?

**Exercise B.** Let $X$ be a space with at least two points and $\Sigma X = (X \times [0,1])/{\sim}$, where $(x,t) \sim (x',t')$ if and only if they are equal or $t = t' = 0$ or $t = t' = 1$, be its unreduced suspension. Show that the projection $\Sigma X \to [0,1]$ is a Hurewicz fibration but not a fibre bundle. *Hint: it's easiest to use the viewpoint of finding a section of the map* $\mathrm{Path}(\Sigma X) \to N_p$, *where* $p$ *is the projection.*

**Exercise C.** Let $p\colon E \to B$ be a fibration, choose a basepoint $*_E$ for $E$ and define $*_B = p(*_E)$. Write $i\colon F = p^{-1}(*_B) \hookrightarrow E$ for the inclusion and choose $*_E$ to be the basepoint for $F$. Show that the natural action $\pi_1(F) \to \mathrm{Aut}(\pi_n(F))$ factors through $i_*\colon \pi_1(F) \to \pi_1(E)$. Deduce that $F$ is simple if $E$ is simply-connected.

**Exercise D.** Let $Z$ be a countably infinite set equipped with the cofinite topology, in other words the topology whose closed sets are all finite subsets of $Z$, together with $Z$ itself.

(a) Show that $Z$ is compactly generated. (*Hint:* how many closed sets does $Z$ have?)

(b) * Show that $Z$ is not path-connected.

(c) Show that the only reflexive relation on $Z$ that is closed as a subset of $Z \times Z$ is the trivial one in which $z \sim \bar{z}$ for all $z, \bar{z} \in Z$.

(d) Deduce that $hZ = \{*\}$ and therefore that $Z$ is not homotopy equivalent to $hZ$.

**Exercise E.** There are several other variations on the concept of Serre/Hurewicz fibration. For example, a map $p\colon E \to B$ has the *weak homotopy lifting property* (WHLP) with respect to a space $Y$ if for all commutative squares

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\;f\;} & E \\ {\scriptstyle \mathrm{id}\times\mathrm{incl}}\downarrow & & \downarrow{\scriptstyle p} \\ Y \times [0,1] & \xrightarrow[\;g\;]{} & B, \end{array} \tag{1}$$

there is a map $G\colon Y \times [0,1] \to E$ such that $p \circ G = g$ and $G(-,0)$ is fibrewise homotopic to $f$ as a map of spaces over $B$. A *Dold fibration* is a map that has the WHLP with respect to all spaces $Y$, and a *Dold-Serre fibration* is one that has the WHLP with respect to all cubes $Y = [0,1]^n$ for $n \geqslant 0$.

(a) Show that the WHLP for $Y$ is equivalent to the statement that there exists a lifting $G: Y \times [0,1] \to E$ making both triangles commute for each commutative square (1) in which $g$ is *initially constant* (meaning that there exists $\epsilon > 0$ such that $g(y,t) = g(t,0)$ for all $(y,t) \in Y \times [0,\epsilon]$). This is sometimes known as the *delayed homotopy lifting property*.

(b) Give an example of a Serre fibration that is not a Dold fibration. (*Hint:* Form the mapping cylinder $M$ of the map $\mathbb{Q}^\delta \to \mathbb{Q}$ from the rationals with the discrete topology to the rationals with the subspace topology from $\mathbb{R}$ and consider the projection $M \to [0,1]$.)

Another related notion is that of a *Serre microfibration*. A map $p: E \to B$ has the *microscopic homotopy lifting property* (MHLP) with respect to $Y$ if, for all commutative squares (1), there is some $\epsilon > 0$ and a map $G: Y \times [0,\epsilon] \to E$ such that $p \circ G = g|_{Y \times [0,\epsilon]}$ and $G(y,0) = f(y)$ for all $y \in Y$. A *Serre microfibration* is a map that has the MHLP with respect to all cubes $Y = [0,1]^n$ for $n \geqslant 0$.

This is a much weaker property than being a Serre fibration, and for many examples it is much easier to check (even for examples that are in fact Serre fibrations). A Serre microfibration $p: E \to B$ does not in general induce a long exact sequence of homotopy groups (in contrast to Serre fibrations, and more generally Dold-Serre fibrations, which do induce such an exact sequence). However, it does have the following property (which is a consequence of the induced long exact sequence in the case of Serre or Dold-Serre fibrations): if $p^{-1}(b)$ is $n$-connected for all $b \in B$ then the map $p$ is $(n+1)$-connected (this means by definition that its homotopy fibres are all $n$-connected, equivalently that it induces isomorphisms on $\pi_i$ for $i \leqslant n$ and surjections on $\pi_{n+1}$ for all choices of basepoints).

(c) Give an example of a map $p: E \to B$ that is a quasi-fibration and a Dold fibration but not a Serre microfibration. (*Hint:* As in exercise 3 on sheet 4, consider the trivial bundle $[0,1]^2 \to [0,1]$ and restrict to a certain subspace of the square.)

(d) Give an example of a map $p: E \to B$ that is a Serre microfibration but not a quasi-fibration.

(e) * Let $p: E \to B$ be a map that is both a Serre microfibration and a Dold-Serre fibration. Prove that $p$ is a Serre fibration.

**Exercise F.** In this exercise we will construct an essential weak phantom map[1] $X \to K(\mathbb{Z},3)$ for a certain CW-complex $X$, in other words – via the representability theorem of Brown – an *essential weak phantom cohomology class* $\alpha \in H^3(X;\mathbb{Z})$, i.e., a non-zero cohomology class that vanishes when restricted to any finite subcomplex.

Let $X$ be the *mapping telescope* of the sequence of maps $S^2 \to S^2 \to S^2 \to \cdots$, where each map in the sequence is equal to a fixed map $f: S^2 \to S^2$ of degree $d \in \mathbb{Z}$

[1] A map is *weak phantom* if it becomes nullhomotopic on any finite subcomplex. It is also *essential* if it is not itself nullhomotopic.

with $|d| \geqslant 2$. In other words, $X$ is the quotient of the disjoint union

$$\coprod_{n \in \mathbb{N}} \big(\{n\} \times S^2 \times [0,1]\big)$$

by the equivalence relation generated by $(n, x, 1) \sim (n+1, f(x), 0)$ for all $n \in \mathbb{N}$ and $x \in S^2$.

(a) Describe a structure of a 3-dimensional CW-complex on $X$ having countably infinitely many $n$-cells for $n \in \{0, 1, 2, 3\}$.

(b) Write down the corresponding cellular chain complex of $X$ and its dual.

(c) Use this to calculate that $H_3(X; \mathbb{Z}) = 0$, but $H^3(X; \mathbb{Z}) \neq 0$.

(d) Show that every element $\alpha \in H^3(X; \mathbb{Z})$ is a weak phantom class.

(e) Similarly, let $X_k$ be constructed as above, using $S^k$ in place of $S^2$ (for any $k \geqslant 1$). Show that $H^{k+1}(X_k; \mathbb{Z}) \neq 0$ and that every element of this cohomology group is a weak phantom class.

**Exercise G.** Suppose we have a pullback square of topological spaces

$$\begin{array}{ccc} A & \xrightarrow{\ h\ } & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\ k\ ]{} & D \end{array}$$

where $f$ and $k$ are both Serre fibrations and $k$ is surjective. Prove that $g$ is also a Serre fibration. Does this implication also hold with "Serre fibration" replaced with "Hurewicz fibration"?

**Exercise H.** Recall from lecture 27 (Monday 28 January 2019) the *join* $\bigstar_{i \in I} X_i$ of a collection $\{X_i \mid i \in I\}$ of topological spaces.

(a) If $A$ is path-connected and $B$ is non-empty, show that $A \star B$ is simply-connected.

(b) Suppose that $A_1, \ldots, A_k$ are non-empty spaces, where $A_i$ is $(n_i - 2)$-connected. Prove that $\bigstar_{i=1}^{k} A_i$ is $(\sum_{i=1}^{k} n_i - 2)$-connected.

(c) Deduce that if $A$ is non-empty and $I$ is infinite, then the *join power* $A^{\star I} = \bigstar_{i \in I} A$ is weakly contractible.