Extra exercises (optional!)

Some optional extra exercises, not for handing in.

Exercise A. Let X and Y be compact, connected manifolds with empty boundary, with X orientable and Y non-orientable, such that $\pi_1(X) \cong \pi_1(Y)$ and this group is finite. Use these to construct two non-homotopy-equivalent spaces with isomorphic homotopy groups. Why does this not contradict Whitehead's theorem?

Exercise B. Let X be a contractible space containing at least two points and let ΣX be its unreduced suspension $(X \times [0,1])/{\sim}$, where $(x,t) \sim (x',t')$ if and only if they are equal or $t = t' = 0$ or $t = t' = 1$. Show that the projection $\Sigma X \to [0,1]$ is a Hurewicz fibration but not a fibre bundle. *Hint: it's easiest to use the viewpoint of finding a section of the map* $\mathrm{Path}(\Sigma X) \to N_p$*, where* p *is the projection* $\Sigma X \to [0,1]$.

Exercise C. Let $p\colon E \to B$ be a fibration, choose a basepoint $*_E$ for E and define $*_B = p(*_E)$. Write $i\colon F = p^{-1}(*_B) \hookrightarrow E$ for the inclusion and choose $*_E$ to be the basepoint for F. Show that the natural action $\pi_1(F) \to \mathrm{Aut}(\pi_n(F))$ factors through $i_*\colon \pi_1(F) \to \pi_1(E)$. Deduce that F is simple if E is simply-connected.

Exercise D. Let Z be a countably infinite set equipped with the cofinite topology, in other words the topology whose closed sets are all finite subsets of Z, together with Z itself.

(a) Show that Z is compactly generated. (*Hint:* how many closed sets does Z have?)

(b) * Show that Z is not path-connected.

(c) Show that the only reflexive relation on Z that is closed as a subset of $Z \times Z$ is the trivial one in which $z \sim \bar{z}$ for all $z, \bar{z} \in Z$.

(d) Deduce that $hZ = \{*\}$ and therefore that Z is not homotopy equivalent to hZ.

Exercise E. There are several other variations on the concept of Serre/Hurewicz fibration. For example, a map $p\colon E \to B$ has the *weak homotopy lifting property* (WHLP) with respect to a space Y if for all commutative squares

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\;f\;} & E \\ {\scriptstyle \mathrm{id} \times \mathrm{incl}}\downarrow & & \downarrow{\scriptstyle p} \\ Y \times [0,1] & \xrightarrow[\;g\;]{} & B, \end{array} \tag{1}$$

there is a map $G\colon Y \times [0,1] \to E$ such that $p \circ G = g$ and $G(-,0)$ is fibrewise homotopic to f as a map of spaces over B. A *Dold fibration* is a map that has the WHLP with respect to all spaces Y, and a *Dold-Serre fibration* is one that has the WHLP with respect to all cubes $Y = [0,1]^n$ for $n \geqslant 0$.

(a) Show that the WHLP for Y is equivalent to the statement that there exists a lifting $G\colon Y \times [0,1] \to E$ making both triangles commute for each commutative square (1) in which g is *initially constant* (meaning that there exists $\epsilon > 0$ such that $g(y,t) = g(t,0)$ for all $(y,t) \in Y \times [0,\epsilon]$). This is sometimes known as the *delayed homotopy lifting property.*

(b) Give an example of a Serre fibration that is not a Dold fibration. (*Hint:* Form the mapping cylinder M of the map $\mathbb{Q}^\delta \to \mathbb{Q}$ from the rationals with the discrete topology to the rationals with the subspace topology from $\mathbb{R}$ and consider the projection $M \to [0,1]$.)

Another related notion is that of a *Serre microfibration.* A map $p\colon E \to B$ has the *microscopic homotopy lifting property* (MHLP) with respect to Y if, for all commutative squares (1), there is some $\epsilon > 0$ and a map $G\colon Y \times [0,\epsilon] \to E$ such that $p \circ G = g|_{Y\times[0,\epsilon]}$ and $G(y,0) = f(y)$ for all $y \in Y$. A *Serre microfibration* is a map that has the MHLP with respect to all cubes $Y = [0,1]^n$ for $n \geqslant 0$.

This is a much weaker property than being a Serre fibration, and for many examples it is much easier to check (even for examples that are in fact Serre fibrations). A Serre microfibration $p\colon E \to B$ does not in general induce a long exact sequence of homotopy groups (in contrast to Serre fibrations, and more generally Dold-Serre fibrations, which do induce such an exact sequence). However, it does have the following property (which is a consequence of the induced long exact sequence in the case of Serre or Dold-Serre fibrations): if $p^{-1}(b)$ is n-connected for all $b \in B$ then the map p is $(n+1)$-connected (this means by definition that its homotopy fibres are all n-connected, equivalently that it induces isomorphisms on π_i for $i \leqslant n$ and surjections on π_{n+1} for all choices of basepoints).

(c) Give an example of a map $p\colon E \to B$ that is a quasi-fibration and a Dold fibration but not a Serre microfibration. (*Hint:* As in exercise 3 on sheet 4, consider the trivial bundle $[0,1]^2 \to [0,1]$ and restrict to a certain subspace of the square.)

(d) Give an example of a map $p\colon E \to B$ that is a Serre microfibration but not a quasi-fibration.

(e) * Let $p\colon E \to B$ be a map that is both a Serre microfibration and a Dold-Serre fibration. Prove that p is a Serre fibration.

Exercise F. In this exercise we will construct an essential weak phantom map[1] $X \to K(\mathbb{Z},3)$ for a certain CW-complex X, in other words – via the representability theorem of Brown – an *essential weak phantom cohomology class* $\alpha \in H^3(X;\mathbb{Z})$, i.e., a non-zero cohomology class that vanishes when restricted to any finite subcomplex.

Let X be the *mapping telescope* of the sequence of maps $S^2 \to S^2 \to S^2 \to \cdots$, where each map in the sequence is equal to a fixed map $f\colon S^2 \to S^2$ of degree $d \in \mathbb{Z}$

[1] A map is *weak phantom* if it becomes nullhomotopic on any finite subcomplex. It is also *essential* if it is not itself nullhomotopic.

with $|d| \geqslant 2$. In other words, X is the quotient of the disjoint union

$$\coprod_{n\in\mathbb{N}} \big(\{n\} \times S^2 \times [0,1]\big)$$

by the equivalence relation generated by $(n, x, 1) \sim (n+1, f(x), 0)$ for all $n \in \mathbb{N}$ and $x \in S^2$.

(a) Describe a structure of a 3-dimensional CW-complex on X having countably infinitely many n-cells for $n \in \{0, 1, 2, 3\}$.

(b) Write down the corresponding cellular chain complex of X and its dual.

(c) Use this to calculate that $H_3(X;\mathbb{Z}) = 0$, but $H^3(X;\mathbb{Z}) \neq 0$.

(d) Show that every element $\alpha \in H^3(X;\mathbb{Z})$ is a weak phantom class.

(e) Similarly, let X_k be constructed as above, using S^k in place of S^2 (for any $k \geqslant 1$). Show that $H^{k+1}(X_k;\mathbb{Z}) \neq 0$ and that every element of this cohomology group is a weak phantom class.

Exercise G. Suppose we have a pullback square of topological spaces

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\;k\;]{} & D \end{array}$$

where f and k are both Serre fibrations and k is surjective. Prove that g is also a Serre fibration. Does this implication also hold with "Serre fibration" replaced with "Hurewicz fibration"?

Exercise H. Recall from lecture 27 (Monday 28 January 2019) the *join* $\bigstar_{i\in I} X_i$ of a collection $\{X_i \mid i \in I\}$ of topological spaces.

(a) If A is path-connected and B is non-empty, show that $A \star B$ is simply-connected.

(b) Suppose that $A_1, \ldots, A_k$ are non-empty spaces, where A_i is $(n_i - 2)$-connected. Prove that $\bigstar_{i=1}^k A_i$ is $(\sum_{i=1}^k n_i - 2)$-connected.

(c) Deduce that if A is non-empty and I is infinite, then the *join power* $A^{\star I} = \bigstar_{i\in I} A$ is weakly contractible.