Lecture 17: 3 December 2018

5 Quasifibrations and the Dold-Thom theorem

Goal: Understand "difference" between π_* and H_*.
References:

- Dold, Thom: "Quasifaserungen und unendliche symmetrische Produkte", Ann. Math. 67.2 (1958)
- Appendix 4K of Hatcher: Algebraic Topology, Cambridge University Press (2010)

5.1 Symmetric products

Definition 5.1. The n-fold *symmetric product* of a space $X \in \mathrm{CG}$ is

$$SP^n(X) = (X^n)/\Sigma_n = \{\text{unordered tuples}\,[x_1, \ldots, x_n]\}.$$

For $(X, e) \in \mathbf{CG}_*$ based, we have maps $i_n\colon SP^n(X) \to Sp^{n+1}(X)$ that insert a basepoint e. The *infinite symmetric product* is the CG-space

$$SP(X) = \{\text{unordered tuples}\,[x_1, \ldots, x_n] \mid n \geqslant 1\}.$$

equipped with the colimit topology of the $SP^n(X)$.

Lemma 5.2. *These constructions define endofunctors of* CG *respectively* CG_*.

Proposition 5.3. *If* (X, e) *is WH, then so are* $SP^n(X)$ *and* $SP(X)$.

The proof uses two lemmas:

Lemma 5.4. *The quotient map* $X^n \to SP^n(X)$ *is a closed map.*

Lemma 5.5. *Each* $i_n\colon SP^n(X) \to SP^{n+1}(X)$ *is a closed inclusion.*

Example 5.6. For $X = S^2 \cong \mathbb{C} \cup \infty$, we have $SP^n(S^2) \simeq \mathbb{C}P^n$ and $SP(S^2) \simeq \mathbb{C}P^\infty$.

Example 5.7. We'll see in the exercises that $S^1 = SP^1(S^1) \to SP(S^1)$ is an equivalence.

5.2 Commutative topological monoids

Definition 5.8. By a *commutative topological monoid* we mean a strictly associative, strictly commutative and strictly unital monoid CGWH space, with basepoint the unit element. Let CMon be the category with objects the commutative topological monoids and morphisms those maps that preserve the multiplication and the unit.

Example 5.9. $SP(X)$ is a commutative topological monoid with multiplication given by "concatenation" of unordered tuples and with unit the basepoint.

Lemma 5.10. $X \mapsto SP(X)$ *is left adjoint to the forgetful functor* CMon $\to$ CGWH_*. *In other words,* $SP(-)$ *satisfies the universal property of the* free commutative topological monoid*: for all* $M \in$ CMon*, any map of spaces* $f\colon X \to M$ *extends uniquely to a map of monoids* $\hat{f}\colon SP(X) \to M$.

Corollary 5.11. *An explicit homotopy inverse for the inclusion* $S^1 \to SP(S^1)$ *(see Example 5.7) is given by the universal map* $\widehat{\mathrm{id}}$ *obtained from* $\mathrm{id}\colon S^1 \to S^1$.

Definition 5.12. The *weak product* $\tilde{\prod}_i X_i$ of based CGWH spaces is the subspace of the product consisting of those tuples $(x_i)_i$ such that $x_i = e$ for all but finitely many i.

Lemma 5.13. *We have*

$$\tilde{\prod}_{i\in I} X_i \cong \operatorname*{colim}_{J\subset I \text{ finite}} \prod_{j\in J} X_j.$$

Lemma 5.14. *The weak product with componentwise multiplication is the categorical coproduct in* CMon.

Corollary 5.15.

$$SP(\bigvee_i X_i) \cong \prod_i^w SP(X_i)$$

Corollary 5.16. *For any countable set I:*

$$\pi_*(SP(\bigvee_i X_i)) \cong \bigoplus_i \pi_*(SP(X_i))$$

(Also in degree 1 because π_1 of a symmetric product is always abelian.)

Remark 5.17. The statement of Corollary 5.16 is true for arbitrary indexing sets; but we never proved that π_* commutes with colimits along closed inclusions indexed by a well-behaved poset. In the context of Brown representability and uniqueness theorems for homology of CW complexes, the wedge axiom is only needed for countable wedge sums (since CW complexes are only defined in a countable range of dimensions).

5.3 The Dold-Thom theorem

Convention: Write $\mathrm{CW}_{*,c}$ for the category of connected based CW complexes.

Theorem 5.18. *(Dold-Thom theorem) Consider the functors*

$$h_n(X) := \pi_n(SP(X)),\ n \geqslant 0.$$

For all $A \to X$, there are boundary homomorphisms

$$\partial_n \colon \pi_n(SP(X/A)) \to \pi_{n-1}(SP(A)$$

that make h_* a reduced cohomology theory on the category $\mathrm{CW}_{*,c}$. Moreover, there is a natural isomorphism $h_* \cong \tilde{H}_*(-;\mathbb{Z})$.

Remark 5.19. We can extend h_* to non-connected CW complexes $Y \in \mathrm{CW}_*$ by setting $\hat{h}_n(Y) = h_{n+1}(\Sigma Y)$. It follows immediately that $h_* \cong H_*(-;\mathbb{Z})$ on CW_*.

Proof. We first verify the homology axioms and then cnstruct the comparison map:
Exactness: We'll show in §5.5 that every $A \to X$ gives rise to a long exact sequence

$$\ldots \to h_n(A) \to h_n(X) \to h_n(X/A) \xrightarrow{\partial_n} h_{n-1}(A) \to \ldots$$

Suspension isomorphism: Follows from previous statement applied to $X \to CX \to \Sigma X$.
Wedge axiom: This is Corollary 5.16.
Homotopy-invariance: Starting with a homotopy $H\colon f \simeq g$, we construct for each stage of the colimit a homotopy $\tilde{H}_n\colon SP^n(f) \simeq SP^n(g) \ldots$
Comparison map: We know from Example 5.7 and Corollary 5.11 that the inclusion $S^1 = SP^1(S^1) \to SP(S^2)$ is a homotopy equivalence with inverse $\hat{\mathrm{id}}\colon SP(S^1) \to S^1$. Together with the Hurewicz isomorphism $\pi_1(S^1) \to \tilde{H}_1(S^1;\mathbb{Z})$, this gives an isomorphism

$$\phi\colon h_1(S^1) \to \tilde{H}_1(S^1).$$

Similar to the proofs of the uniqueness statements that we have seen previously, ϕ extends to a natural transformation of homology theories that is an isomorphism for all spheres S^n, $n \geqslant 1$, hence for all connected CW complexes. □

Remark 5.20. The composite

$$\phi \circ \pi_n(\mathrm{incl})\colon \pi_n(X) \to \pi_n(SP(X)) \to \tilde{H}_n(X)$$

is the Hurewicz homomorphism, up to isomorphism. The Hurewicz homomorphism is natural and compatible with suspension isomorphisms, so it suffices to check the statement for $X = S^1$, where it holds by construction of ϕ.

5.4 Applications

Definition 5.21. X is called a *Moore space* of type $M(G, n)$ if $\tilde{H}_n(X;\mathbb{Z}) = G$ and $\tilde{H}_i(X;\mathbb{Z}) = 0$ for $i \neq n$. If $n > 1$, we require that $M(G, n)$ be simply-connected.

Corollary 5.22. *The functor SP sends Moore spaces of type $M(G, n)$ to Eilenberg-MacLane spaces of type $K(G, n)$.*

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Corollary 5.23. *If a connected $X \in \mathrm{CW}_{*,c}$ admits the structure of a commutative topological monoid, then X is weakly equivalent to a product of Eilenberg-MacLane spaces.*

The proof needs a lemma.

Lemma 5.24. *Let X be a based space such that $\pi_1(X)$ is abelian. For all $n \geqslant 1$, there exists a space of type $M(\pi_n(X), n)$, simply connected if $n > 1$, together with a map $f_n\colon M(\pi_n(X), n) \to X$ that induces an isomorphism on π_n.*

5.5 Quasifibrations

Left to show: exactness of $h_* = \pi_*(SP(-))$, which follows from:

Theorem 5.25. *If $f\colon A \to Z$ is a map of connected CW complexes with mapping cylinder M_f, then the map $SP(M_f) \to SP(M_f/A)$ induced by $M_f \to M_f/A$ is a quasifibration with fibre $SP(A)$, hence*

$$SP(A) \to SP(M_f) \to SP(M_f/A)$$

gives rise to a long exact sequence of homotopy groups.

Example 5.26. Let $f\colon A \to X$ be a map of based connected CW complexes. Pick a path γ in M_f such that $\gamma(t) \in X - A$ for $t > 0$ and $\gamma(0) = a$ is not the basepoint. Then $p(\gamma)$ is a path in $SP(M_f/A)$. Any lift $\tilde{\gamma}$ to $SP(M_f)$ would have to satisfy $\tilde{\gamma}(0) \in a \cdot SP(A)$, so there is no lift starting at $e \in SP(M_f)$.

Recall the notion of a quasifibration from Definition 2.60.

Lemma 5.27. *For Y path-connected, a map $p\colon X \to Y$ is a quasifibration if and only if*

$$p_*\colon \pi_*(X, p^{-1}(y), x) \to \pi_*(Y, y)$$

is an isomorphism for all $y \in Y, x \in p^{-1}(y), i \geqslant 0$.

Lemma 5.28 (Long exact sequence of homotopy groups of a triple)**.** *Let (X, A, B, x) be a triple (i.e. $x \in B \subseteq A \subseteq X$). Then there is a long exact sequence (of abelian groups for $n \geqslant 3$, groups for $n \geqslant 2$, pointed sets otherwise) of the form*

$$\ldots \to \pi_n(A, B, x) \to \pi_n(X, B, x) \to \pi_n(X, A, x) \to \pi_{n-1}(A, B, x) \to \ldots$$

Proof. Similar to the proof for a pair. □

Proposition 5.29. *Let $f\colon (X;U_1,U_2) \to (Y;V_1,V_2)$ be a map of triads such that*

$$f_*\colon \pi_*(U_i, U_1 \cap U_2) \to \pi_*(V_i, V_1 \cap V_2)$$

are isomorphisms for $i = 1,2$ and all choices of basepoints, then so are $f_*\colon \pi_*(X,U_i) \to \pi_*(Y,V_i)$.

Proof. Omitted. This is a relative version of Theorem 3.15. The proof is very technical and can be found e.g. as Proposition 4K.1 in Hatcher's book. □

Lemma 5.30. *Let Y be path-connected. The map $p\colon X \to Y$ is a quasifibration if one of the following is satisfied:*

(i) All fibres of p are path-connected and there is a triad $(Y;V_1,V_2)$ such that each $p^{-1}(V_i) \to V_i$ and $p^{-1}(V_1 \cap V_2) \to V_1 \cap V_2$ are quasifibrations.

(ii) Y is the colimit of path-connected spaces Y_i along closed inclusions $Y_i \to Y_{i+1}$, such that each $p^{-1}(Y_i) \to Y_i$ is a quasifibration.

(iii) There are subspaces $\tilde{U} \subseteq X$, $U \subseteq Y$ and a commutative diagram of deformations

$$\begin{array}{ccc} I \times X & \xrightarrow{\tilde{h}} & X \\ {\scriptstyle I\times p}\downarrow & & \downarrow{\scriptstyle p} \\ I \times Y & \xrightarrow[h]{} & Y \end{array}$$

with

$$\tilde{h}_0 = \mathrm{id}_X,\ \tilde{h}_t(\tilde{U}) \subseteq \tilde{U},\ \tilde{h}_1(X) \subseteq \tilde{U},$$

$$h_0 = \mathrm{id}_Y,\ h_t(U) \subseteq U,\ h_1(Y) \subseteq U,$$

such that $p\colon \tilde{U} \to U$ is a quasifibration and, for all $y \in Y$,

$$\tilde{h}_1 : p^{-1}(y) \to p^{-1}(h_1(y))$$

is a weak equivalence.

Proof of Theorem 5.25. We have to show: $X = SP(M_f) \to Y = SP(M_f/A)$ is a quasifibration. Set $Y_0 = *$, $Y_n = SP^n(M_f/A)$ and $X_n = p^{-1}(Y_n)$. Thus X_n is the space of unordered finite tuples of elements of M_f with at most n entries contained in $M_f - A$.
Step 1: By part (ii) of Lemma 5.30, it is enough to show that each $X_n \to Y_n$ is a quasifibration. We will prove this by induction on n, the case $n = 0$ being trivial.
Step 2: We write $Y_n = U \cap (Y_n - Y_{n-1})$, where U is an open neighbourhood of Y_{n-1} in Y_n that will be constructed later. By part (i) of Lemma 5.30, it suffices to show that all fibres of $p\colon X_n \to Y_n$ are path-connected and that p is a quasifibration over $Y_n - Y_{n-1}$, over U, and over the intersection.
Step 3: We show that all fibres are connected: Let $y \in Y_n$. If $y = [e]$, then $p^{-1}(y) = SP(A)$. If $y \neq e$, then it can be written in a unique way as a product of elements of M_f/A. Let $\tilde{y}$ be the unique lift to $SP(M_f - A)$ of the product of those factors of y that are not the basepoint of M_f/A. Then $p^{-1}(y)$ is the coset $SP(A) \cdot \tilde{y}$, hence connected.
Step 4: We show that p is a trivial fibre bundle over $Y_n - Y_{n-1}$. It follows that it is a quasifibration over any subspace of $Y_n - Y_{n-1}$, in particular over $(Y_n - Y_{n-1}) \cap U$. (…details given in lecture…)[1]
Step 5: We construct U geometrically and show that p is a quasifibration over U. (…details given in lecture…) □

[1] Addendum to the proof of the lecture: At some point in the proof, we used without justification that $SP(M_f)\cdot V$ is closed in $SP(M_f)$ if V is a closed subset of $SP(A)$. This is indeed true and can be seen as follows: Since $A \subseteq M_f$ is closed, $V \subseteq SP(M_f)$ is closed. It now suffices to check that the multiplication map $SP(M_f) \times SP(M_f) \to SP(M_f)$ is a closed map. This can be verified by restricting to $SP^i(M_f) \times SP^j(M_f)$, where it can be proven by mimicking the proof of Lemma 5.5.