Algebraic Topology I

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Abstract

An outline of the topics that I covered in the lecture course *Algebraic Topology I* between October 2018 and January 2019. Webpage: mdp.ac/teaching/18-algebraic-topology.html.

Contents

0. Topic index

NB: This is only a partial list of key topics in the lectures.

1. Monday 8 October

1. Review

1.1 Definition of $\pi_n(X)$.

1.2 Basic properties of $\pi_n(X)$:

- It is a group for $n \geqslant 1$.
- It is abelian for $n \geqslant 2$.
- There is a natural action of $\pi_1(X)$ on $\pi_n(X)$ for all $n \geqslant 1$.
- π_n is a 2-functor $\mathrm{Top}_* \longrightarrow \mathrm{Grp}$, in particular it takes homotopic maps to the same homomorphism, and therefore homotopy equivalences to isomorphisms.
- π_n preserves products.

1.3 Definition of H-spaces, as well as H-monoid, H-group and homotopy-commutative H-group. Dualising these definitions by reversing the arrows and replacing $\times$ with the categorical coproduct in Top_*, the wedge $\vee$, we get the dual notions of co-H-space, etc.

1.4 Examples:

- Topological groups are H-groups.
- S^n is a co-H-group for $n \geqslant 1$, and is homotopy-commutative for $n \geqslant 2$.
- More generally: Definition of the *reduced suspension* ΣX of a based space X. Exercise: ΣX is a co-H-group and $\Sigma(\Sigma X)$ is a homotopy-commutative co-H-group.
- Definition: $\Omega X = \mathrm{Maps}_*(S^1, X)$ with the compact-open topology is the *loopspace* of X. Exercise: ΩX is an H-group and $\Omega(\Omega X)$ is a homotopy-commutative H-group.

1.5 Lemma: if X is an H-space, then $\pi_n(X)$ is abelian also for $n = 1$ and, writing the operation on $\pi_n(X)$ as $+$, we have the formula,

$$[f] + [g] = [m \circ (f, g)],$$

where $m \colon X \times X \to X$ is the H-space structure of X.

1.6 Definition: if the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial for all $n \geqslant 1$, then X is called a *simple* space. For any other property $\mathcal{P}$ of group actions on groups, if the action of $\pi_1(X)$ on $\pi_n(X)$ is $\mathcal{P}$ for all $n \geqslant 1$, then X is called a $\mathcal{P}$ space. For example, $\mathcal{P} =$ nilpotent is an important example.

1.7 If $f \colon X \to Y$ is a covering map, then $\pi_n(f)$ is an isomorphism for all $n \geqslant 2$.

1.8 Definition of relative homotopy groups $\pi_n(X, A)$.
When $n \geqslant 2$ this has a group structure, and when $n \geqslant 3$ it is abelian.

1.9 Proposition: Associated to any based pair of spaces (X, A), there is a natural long exact sequence of the form

$$\cdots \to \pi_n(A) \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, A) \longrightarrow \pi_{n-1}(A) \to \cdots,$$

where the map $\pi_n(X, A) \to \pi_{n-1}(A)$ is given by $[f] \mapsto [f|_{I^{n-1}}]$. Moreover, the group $\pi_1(A)$ acts on each space in the sequence and the horizontal maps are equivariant with respect to these actions.

1.10 Definition: n-connectedness.

1.11 Remark: how to define $\pi_0(X, A)$ in order for the end of the long exact sequence above to really be exact; we define it as

$$\pi_0(X, A) = \pi_0(X/A) = \{\text{path-components of } X \text{ disjoint from } A\} \cup \{\{\text{the rest of } X\}\}.$$

So the pair (X, A) is 0-connected if and only if each path-component of X contains a point of A.

1.12 Definition:

- CW-complex X
- geometric realisation $|X|$ of a CW-complex X
- subcomplex of a CW-complex
- Note: if A is a subcomplex of a CW-complex X, then $|A|$ is naturally a subspace of $|X|$. Moreover, any subspace $S \subseteq |X|$ can be equal to $|A|$ for at most one subcomplex A of X, in other words, the subspace $|A| \subseteq |X|$ determines the subcomplex $A \subseteq X$.
- CW-pair

1.13 The Cellular Approximation Theorem: any continuous map between (geometric realisations of) CW-complexes is homotopic to a *cellular map*, that is, a map taking the n-skeleton of its domain into the n-skeleton of its codomain, for all $n \geqslant 0$. If the map is already cellular on a subcomplex, then this homotopy can be taken to be constant on that subcomplex.

1.14 Definition: weak homotopy equivalence.

1.15 CW-approximation theorem:

(a) For any space X, there is a CW-complex Y and a weak equivalence $|Y| \to X$.

(b) If A is a CW-complex, X is a space with $|A| \subseteq X$ and $n \geqslant 0$ is an integer, then there exists a CW-complex Y containing A as a subcomplex and a factorisation of the inclusion $|A| \hookrightarrow X$ into

$$|A| \xhookrightarrow{i} |Y| \xrightarrow{f} X,$$

where i just denotes the inclusion $|A| \subseteq |Y|$, such that
- $\pi_k(i)$ is an isomorphism for $k < n$,
- $\pi_n(i)$ is a surjection and $\pi_n(f)$ is an injection,
- $\pi_n(f)$ is an isomorphism for $k > n$.

- Note: (b) implies (a).

2. Wednesday 10 October

1.16 Examples of CW-complexes:
- All simplicial complexes (and therefore all smooth manifolds).
- $\mathbb{R}^n$, S^n, $\mathbb{RP}^n$, $\mathbb{CP}^n$.
- Compact surfaces (picture for the closed, connected, orientable surfaces of genus 2).
- Quotients: if (X, A) is a CW-pair, then X/A is a CW-complex.
(Here, I am using the usual abuse of notation by conflating a CW-complex with its geometric realisation. A formally correct statement is that the quotient space $|X|/|A|$ admits a canonical CW-structure.)
- Products: if X and Y are CW-complexes, then so is $X \times Y$.
(There is a subtlety here that I skipped over in the lectures: this statement is true only as long as we modify the topology on the product slightly. For any space X, there is another topology on X, called the compactly-generated topology and denoted by X_c, such that (a) the identity map $X_c \to X$ is continuous and (b) the two topologies on X have the same compact subspaces. The correct version of the above statement is then the following: if X and Y are (geometric realisations of) CW-complexes, then so is $(X \times Y)_c$. The properties (a) and (b) of the compactly-generated topology, and the fact that geometric realisations of CW-complexes are Hausdorff, imply that the identity map $(X \times Y)_c \to X \times Y$ is a weak equivalence. We are therefore not changing too much by modifying the topology of the product in this way.)
- Attaching spaces: if (X, A) is a CW-pair and $f\colon A \to Y$ is a cellular map, then

$$X \cup_f Y = (X \sqcup Y)/(a \sim f(a) \text{ for all } a \in A)$$

is a CW-complex.
- Self-attaching spaces: if A and B are subspaces of X and $f\colon A \to B$ is continuous, then

$$X/f = X/(a \sim f(a) \text{ for all } a \in A).$$

If X is a CW-complex, A and B are subcomplexes and f is cellular, then X/f is a CW-complex.

1.17 Definition: finite / finite-dimensional CW-complexes.

1.18 Definition: cofibrations.

1.19 Lemma: if (X, A) is a CW-pair, then the inclusion $A \hookrightarrow X$ is a cofibration.

1.20 Constructions on spaces:
- Reduced join.
- Smash product.
- Wedge sum.

- Loopspace.

1.21 Proposition: if X and Y are based spaces, C is a co-H-group and G is an H-group, then
  (a) $\langle C, Y\rangle$ and $\langle X, G\rangle$ are groups.
  (b) In particular, $\langle \Sigma X, Y\rangle$ and $\langle X, \Omega Y\rangle$ are groups.
  (c) There is a group isomorphism $\langle \Sigma X, Y\rangle \;\cong\; \langle X, \Omega Y\rangle$.

\- Proof.

1.22 Construction: mapping cylinders. (Note: if $f \simeq g$ then $M_f \simeq M_g$.)

1.23 Theorem (Whitehead): if X and Y are connected CW-complexes and $f\colon X \to Y$ is a weak equivalence, then it is a homotopy equivalence.

1.24 Lemma ("Compression Lemma"): Let (X, A) be a CW-pair and (Y, B) any pair of non-empty spaces. If there is an n-cell of X that is not in A, assume that $\pi_n(Y, B) = 0$ for all basepoints of B. Then any map $(X, A) \to (Y, B)$ is homotopic rel. A to a map with image contained in B.

\- Proof.

3. Monday 15 October

1.25 Proposition (Whitehead, special case): Suppose that (X, A) is a CW-pair and that X and A are both path-connected. If the inclusion $A \hookrightarrow X$ is a weak equivalence, then there is a strong deformation retraction of X onto A.

\- Proof.

\- Proof of Theorem 1.23.

1.26 Theorem: if $f\colon X \to Y$ is a weak equivalence, then it induces isomorphisms on homology and cohomology.

\- Proof.

1.27 Proposition: if $f\colon X \to Y$ is a weak equivalence, then it induces bijections $[A, X] \to [A, Y]$ and $\langle A, X\rangle \to \langle A, Y\rangle$ for all based CW-complexes A.

1.28 Definition (Hurewicz homomorphism): $h\colon \pi_n(X, A) \to H_n(X, A)$.

1.29 Lemma: h is well-defined, a group homomorphism for $n \geqslant 2$ and invariant under the action of $\pi_1(A)$.

1.30 Theorem (Hurewicz): Suppose that (X, A) is n-connected for $n \geqslant 1$ and that X and A are both 0-connected. Then
  (a) $H_k(X, A) = 0$ for $k \leqslant n$,
  (b) $h\colon \pi_{n+1}(X, A) \to H_{n+1}(X, A)$ is surjective and induces an isomorphism

$$\pi_{n+1}(X, A)/\pi_1(A) \;\cong\; H_{n+1}(X, A).$$

1.31 Theorem (the "homology Whitehead theorem"): Suppose that X and Y are 0-connected, simple CW-complexes and let $f\colon X \to Y$ be a continuous map that induces isomorphisms on H_n for all $n \geqslant 0$. Then f is a homotopy equivalence.

\- Proof, modulo one lemma that will be proven in a few weeks' time. (The lemma is not necessary if we make the stronger assumption that X and Y are 1-connected.)

2. Fibrations and cofibrations

2.1 Definition: homotopy extension property (HEP).

2.2 Definition: cofibrations.

2.3 Lemma: $i\colon A \to X$ is a cofibration if and only if it has the HEP for its mapping cylinder M_i, if and only if the canonical map $M_i \to X \times [0, 1]$ has a retraction.

\- Proof.

2.4 Proposition: if (X, A) is a CW-pair, then the inclusion $A \hookrightarrow X$ is a cofibration.

2.5 Definition: NDR-pair.

2.6 Proposition: an inclusion $A \hookrightarrow X$ is a closed cofibration if and only if (X, A) is an NDR-pair.

4. Wednesday 17 October

2.4 Proposition: if (X, A) is a CW-pair, then the inclusion $A \hookrightarrow X$ is a cofibration.

2.4.1 Sublemma: If $i\colon A \to X$ is an inclusion, write M_i for its mapping cylinder (equipped with the colimit topology from its definition as a pushout, i.e., the weak topology generated by

the subspaces $A \times [0,1]$ and $X \times \{0\}$) and set $C = (A \times [0,1]) \cup (X \times \{0\}) \subseteq X \times [0,1]$ with the subspace topology. The universal property of the pushout gives us a continuous bijection $f \colon M_i \to C$.
If there exists a retraction of the inclusion $C \hookrightarrow X \times [0,1]$, then f is a homeomorphism. For a proof, see pages 532–533 of [Hatcher].

- Consequence: if there exists retraction of the inclusion $C \hookrightarrow X \times [0,1]$, then i is a cofibration.
- Proof of Proposition 2.4, using this point-set topological fact.

2.5 Definition: NDR-pair. Note that, in particular, if (X, A) is an NDR-pair, then A must be closed in X and admit an open neighbourhood that strongly deformation retracts onto it.

2.6 Proposition: Let A be a closed subset of X. Then $A \hookrightarrow X$ is a cofibration if and only if (X, A) is an NDR-pair.

- Proof.

2.7 Example: if $f \colon X \to Y$ is a map, then the inclusion $X \hookrightarrow M_f$ is an NDR-pair, and therefore a cofibration. (It is also not hard to check directly from the definition that it is a cofibration.)

- So any continuous map has a factorisation as a cofibration followed by a homotopy equivalence. We will see soon that this is, in a precise sense, unique.

2.8 Lemma: if $i \colon A \to X$ is a cofibration and we let Y be the pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d] \\ X \arrow[r] & Y, \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

then the map $B \to Y$ is also a cofibration.

2.9 Proposition: Suppose that we have a diagram

$$\begin{tikzcd} & X \arrow[d, "f"] \\ A \arrow[ur, hook, "i"] \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

where i and j are cofibrations and f is a homotopy equivalence. Then there exists a map $g \colon Y \to X$ such that $gj = i$ and homotopies $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$ rel. A.

- Reinterpretation: let A/Top be the $(2,1)$-category whose objects are continuous maps $A \to X$, whose morphisms are continuous maps $X \to Y$ making the triangle with the two maps from A commute, and whose 2-morphisms are homotopies $H \colon f \simeq g \colon X \to Y$ such that $H(-,t)$ makes the triangle commute for all times t. This is the category of *spaces under* A. Since it is a 2-category, it has a notion of *homotopy equivalence in* A/Top. There is a forgetful functor
$$F \colon A/\mathrm{Top} \longrightarrow \mathrm{Top}$$
that remembers just the codomain of a map $A \to X$. Proposition 2.9 then says the following. If f is a morphism in A/Top between cofibrations and $F(f)$ is a homotopy equivalence in Top, then f is a homotopy equivalence in A/Top.
- Proof: next lecture.

2.10 Proposition: Let Map be the $(2,1)$-category whose objects are all continuous maps between spaces, whose morphisms are commutative squares, and whose 2-morphisms are pairs of homotopies H, K such that $H(-,t)$ and $K(-,t)$ make the square commute for all times t. There are 2-functors

$$\begin{tikzcd} & & \mathrm{Top} \\ A/\mathrm{Top} \arrow[r, hook] \arrow[rr, bend right, "F"'] & \mathrm{Map} \arrow[ur, "\mathrm{dom}"] \arrow[r, "\mathrm{codom}"] & \mathrm{Top} \end{tikzcd}$$

If (f, g) is a morphism in Map between cofibrations and f and g are both homotopy equivalences in Top, then (f, g) is a homotopy equivalence in Map.[1]

[1] Note: this statement does *not* include Proposition 2.9 as a special case. If we are in the situation of Proposition 2.9, we may apply Proposition 2.10 to the pair (f, id_A) and deduce that (f, id_A) is a homotopy equivalence in Map. But this is a weaker statement than f being a homotopy equivalence in A/Top. This is essentially because the inclusion $A/\mathrm{Top} \hookrightarrow \mathrm{Map}$ is not surjective on 2-hom sets.

2.11 Corollary: From Construction 1.22 and Example 2.7, we know that, for any map $f\colon X \to Y$, there exists a factorisation of f as $X \to Z \to Y$, where the first map is a cofibration and the second is a homotopy equivalence. This is unique, in the following sense. If we have a commutative diamond

$$\begin{tikzcd} & Z \arrow[dr, "\pi_1"] & \\ X \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & Y \\ & Z' \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

in which i_1, i_2 are cofibrations and π_1, π_2 are homotopy equivalences, then there is a map $h\colon Z \to Z'$ that is a homotopy equivalence in X/Top and such that $\pi_2 h \simeq \pi_1$.

2.12 Definition: The *homotopy cofibre* of $f\colon X \to Y$ is $C_f = Z/i(X) \in \mathrm{Top}_*$, for any factorisation $\pi \circ i\colon X \to Z \to Y$ of f into a cofibration i followed by a homotopy equivalence π.

2.13 Corollary: C_f is well-defined up to homotopy-equivalence in Top_*.

- Proof: apply Corollary 2.11 and the fact that $Q\colon A/\mathrm{Top} \to \mathrm{Top}_*$ (which is defined on objects as $Q(i\colon A \to X) = X/i(A)$) is a 2-functor, and therefore takes homotopy equivalences rel. A to based homotopy equivalences.
- Proof of Corollary 2.11, using Proposition 2.9 and the following:
- General fact: if $ab \simeq c$ with b a cofibration, then there is a homotopy $a \simeq \hat{a}$ such that $\hat{a}b = c$.

5. Monday 22 October

- Correction to proof of one direction of Proposition 2.6 from the last lecture (if A is a closed subspace of X and the inclusion $A \hookrightarrow X$ is a cofibration, then (X, A) is an NDR-pair).

2.14 Facts: if $i\colon A \to X$ is a cofibration, then:

(a) i is a topological embedding (continuous, injective, homeomorphism onto its image) [Exercise on sheet 2]

(b) The canonical map $s\colon M_i \to X \times [0,1]$ is a continuous injection. If A is closed in X or $s(M_i)$ is a retract of $X \times [0,1]$ then s is a topological embedding.

2.15 Lemma: Let $i\colon A \hookrightarrow X$ be an inclusion. Then i is a cofibration if and only if the inclusion $(A \times [0,1]) \cup (X \times \{0\}) \hookrightarrow X \times [0,1]$ admits a retraction.
This follows from Lemma 2.3 and Sublemma 2.4.1.

2.16 Lemma: If $i\colon A \hookrightarrow X$ is a cofibration and Z is any space, then $i \times \mathrm{id}_Z\colon A \times Z \to X \times Z$ is also a cofibration.

2.17 Remark: If Z is locally compact, then the mapping cylinder $M_{i\times \mathrm{id}_Z}$ is homeomorphic to $M_i \times Z$, and we could use M_i (with the pushout topology) instead of $(A \times [0,1]) \cup (X \times \{0\})$ in the proof of Lemma 2.16.

- Proof of Proposition 2.9, in three steps:
  (I) If $A \hookrightarrow X$ is a cofibration and $f\colon X \to X$ is a self-map such that $f|_A = \mathrm{id}_A$ and $f \simeq \mathrm{id}_X$, then f has a *left-inverse up to homology rel. A*, namely a self-map $g\colon X \to X$ such that $g|_A = \mathrm{id}_A$ and $gf \simeq \mathrm{id}_X$ rel. A.
  - Proof: choose a homotopy $H\colon f \simeq \mathrm{id}_X$ and apply the HEP for $A \hookrightarrow X$ to the maps $(\mathrm{id}_X, H|_{A\times[0,1]})$ to obtain a homotopy $K\colon \mathrm{id}_X \simeq g$ such that $g|_A = \mathrm{id}_A$. Then apply the HEP for $A \times [0,1] \hookrightarrow X \times [0,1]$ (by Lemma 2.16) to some carefully-constructed maps to obtain a homotopy $gf \simeq \mathrm{id}_X$ rel. A.

  (II) In the setting of Proposition 2.9, f has a left-inverse up to homology rel. A (call it g).
  - Proof: by reduction to the special case in step (I).

  (III) This map g is also a *right*-inverse up to homotopy rel. A for f.
  - Proof: apply step (II) again to g to obtain a left-inverse up to homotopy rel. A for g (call it h), and then note that $h \simeq f$ rel. A.

2.18 Corollary: if the inclusion $i\colon A \hookrightarrow X$ is a cofibration and a homotopy-equivalence, then it is a strong deformation retract.

- Proof: apply Proposition 2.9 to the diagram

$$\begin{tikzcd} & A \arrow[d, hook, "i"] \\ A \arrow[ur, "\mathrm{id}_A"] \arrow[r, hook, "i"'] & X. \end{tikzcd}$$

2.19 Definition: the *homotopy lifting property* (HLP), *fibrations* (a.k.a. *Hurewicz fibrations*) and *Serre fibrations.*

- Remark: there exist maps that are Serre fibrations but not Hurewicz fibrations (see exercise sheet 2).

2.20 Proposition: $p\colon E \to B$ has the HLP for $Y = [0,1]^n$ for all $n \geqslant 0$ if and only if it has the HLP for all CW-complexes Y.

- Many statements for cofibrations dualise to the setting of fibrations, for example:
  - The analogues of Propositions 2.9 and 2.10 hold.
  - For any map $f\colon X \to Y$ there exists a factorisation $f = \pi \circ i\colon X \to Z \to Y$ such that i is a homotopy equivalence and π is a fibration, and this factorisation is essentially unique (*cf.* Corollary 2.11).

  [To be stated more precisely and proved next lecture.]
- An obvious example of a Hurewicz (hence also Serre) fibration is the projection $X \times B \to B$. This generates many more examples, using the following local-to-global principle:
- Theorem: Let $p\colon E \to B$ be a map and $\mathcal{U}$ an open cover of B.
  (a) If $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ is a Serre fibration for all $U \in \mathcal{U}$, then so is p.
  (b) If $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ is a Hurewicz fibration for all $U \in \mathcal{U}$, and $\mathcal{U}$ is numerable, then p is also a Hurewicz fibration.
  [An open cover is *numerable* if it is *locally finite* (every $b \in B$ has an open neighbourhood V such that $U \cap V = \varnothing$ for all but finitely many $U \in \mathcal{U}$) and for each open set $U \in \mathcal{U}$ we have $B \smallsetminus U = \lambda^{-1}(0)$ for some continuous map $\lambda\colon B \to [0,1]$.]
- As a corollary, every *fibre bundle* is a Serre fibration and every *fibre bundle that admits a numerable trivialising open cover* (such as any fibre bundle over a paracompact base space) is a Hurewicz fibration. More on this (including a proof of the above theorem) next time.

6. Wednesday 24 October

2.20 Proposition: $p\colon E \to B$ is a Serre fibration if and only if it has the HLP for all CW-complexes Y if and only if it has the HLP for all CW-pairs (Y, A).

- Here, the HLP for a *pair* of spaces (Y, A) means the right lifting property with respect to the inclusion $(Y \times \{0\}) \cup (A \times [0,1]) \hookrightarrow Y \times [0,1]$.
- Note: the constant map $E \to \{*\}$ is a fibration for any space E, so this proposition includes the statement that the inclusion $A \hookrightarrow Y$ is a cofibration for any CW-pair (Y, A) (this was Proposition 2.4).

2.21 Lemma: compositions and pullbacks of Serre/Hurewicz fibrations are Serre/Hurewicz fibrations.

2.22 Remark: if $\mathcal{C}$ is a class of spaces, we may define $\mathcal{C}$-fibration to mean satisfying the HLP for all spaces in $\mathcal{C}$, and then Lemma 2.21 is true for $\mathcal{C}$-fibrations, for any class $\mathcal{C}$.

2.23 Proposition (dual of 2.9): Suppose that p and q are fibrations and f is a homotopy equivalence in the diagram

$$\begin{tikzcd} X \arrow[dr, "p"] \arrow[dd, "f"'] & \\ & B \\ Y \arrow[ur, "q"'] & \end{tikzcd}$$

Then there is a map $g\colon Y \to X$ such that $pg = q$ and homotopies $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$ over B. This may be rephrased in terms of a 2-functor $\mathrm{Top}/B \to \mathrm{Top}$, as in Proposition 2.9.

- Sketch proof.

2.24 Proposition (dual of 2.10): Suppose that (f, g) is a morphism in Map between fibrations, and f and g are homotopy equivalences in Top. Then (f, g) is a homotopy equivalence in Map.

2.25 Remark: this does *not* contain Proposition 2.23 as a special case – *cf.* footnote on page 7.

2.26 Proposition: For any map $f\colon X \to Y$ there is a factorisation of the form $f = pi\colon X \to Z \to Y$ with i a homotopy equivalence and p a fibration. This is unique in the sense that, if we have a commutative diamond

$$\begin{tikzcd}[row sep=small] & Z \arrow[dr, "p_1"] & \\ X \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & Y \\ & Z' \arrow[ur, "p_2"'] & \end{tikzcd}$$

in which i_1, i_2 are homotopy equivalences and p_1, p_2 are fibrations, then there is a map $h\colon Z \to Z'$ that is a homotopy equivalence in Top/Y and such that $hi_1 \simeq i_2$.

- Proof. For the existence we define N_f to be the subspace of $X \times \mathrm{Path}(Y) = X \times \mathrm{Map}([0,1], Y)$ of pairs (x, γ) such that $f(x) = \gamma(0)$. This is the dual of the mapping cylinder construction. The map f factors through N_f via $x \mapsto (x, \mathrm{const}_{f(x)})$ and $(x, \gamma) \mapsto \gamma(1)$. Then one has to check that these maps are a homotopy equivalence and a fibration respectively. The proof of uniqueness is exactly dual to Corollary 2.11.

2.27 Definition: The *homotopy fibre* of $f\colon X \to Y$ is $F_f = p^{-1}(*_Y) \in \mathrm{Top}$, for any factorisation of f into a homotopy equivalence followed by a fibration p.

- Notes:
  (1) This depends on a choice of basepoint $*_Y$ of Y.
  (2) For comparison: the homotopy cofibre C_f does not depend on a choice of basepoint of Y, and C_f is naturally a based space; the homotopy fibre F_f depends on a choice of basepoint of Y, and F_f is naturally an *unbased* space.
  (3) This definition is independent of the choice of factorisation, up to homotopy equivalence. This follows from the uniqueness statement of Proposition 2.26 and the fact that $F\colon \mathrm{Top}/B \to \mathrm{Top}$ (which, for a based space B, is defined on objects by $F(p\colon X \to B) = p^{-1}(*_B)$) is a 2-functor, and therefore takes homotopy equivalences over B to homotopy equivalences.
  (4) Using our construction of N_f, one explicit model for F_f is the subspace of $X \times \mathrm{Path}(Y)$ of pairs (x, γ) such that $\gamma(0) = f(x)$ and $\gamma(1) = *_Y$.

2.28 Lemma: if the map $p\colon E \to B$ is a fibration and a homotopy equivalence, then is is *shrinkable*: there exists a section $s\colon B \hookrightarrow E$ such that sp is fibrewise homotopic to id_E.

- Proof: apply Proposition 2.23 to the diagram

$$\begin{tikzcd}[row sep=small] E \arrow[d, "p"'] \arrow[dr, "p"] & \\ B \arrow[r, "\mathrm{id}_B"'] & B. \end{tikzcd}$$

2.29 Definition: an open cover $\mathcal{U}$ of B is *numerable* if
  - every $b \in B$ has an open neighbourhood V s.t. $U \cap V \neq \varnothing$ for only finitely many $U \in \mathcal{U}$,
  - for every $U \in \mathcal{U}$ there is a continuous map $\lambda\colon B \to [0,1]$ s.t. $\lambda^{-1}(0) = B \smallsetminus U$.

2.30 Theorem: Let $p\colon E \to B$ be a map and $\mathcal{U}$ an open cover of B.
  (a) If $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ is a Serre fibration for all $U \in \mathcal{U}$, then so is p.
  (b) If $p|_{p^{-1}(U)}\colon p^{-1}(U) \to U$ is a Hurewicz fibration for all $U \in \mathcal{U}$, and $\mathcal{U}$ is numerable, then p is also a Hurewicz fibration.

- Proof of part (a). Given a homotopy lifting problem for $[0,1]^n \times \{0\} \hookrightarrow [0,1]^{n+1}$, pull back the open cover $\mathcal{U}$ to the $(n+1)$-cube and subdivide it into a grid of N^{n+1} subcubes of edge-length $\frac{1}{N}$ such that each subcube is mapped into $U \subseteq B$ for some $U \in \mathcal{U}$. This is possible for some N by the Lebesgue lemma. Then lift the homotopy over these subcubes, one at a time (using Proposition 2.20).

2.31 Definition: if $\mathcal{P}$ is a property of topological spaces, we say that $X \in \mathrm{Top}$ is *locally* $\mathcal{P}$ if:
  - for any $x \in X$ and open neighbourhood U of x, there exists another open neighbourhood V of x and a subset $V \subseteq A \subseteq U$ such that A has property $\mathcal{P}$.

2.32 Definition: for spaces X and Y, let $\mathrm{Map}(X, Y)$ to be the set of continuous maps $X \to Y$ with the *open-open topology*. This is the topology generated by the subbasis

$$\{B(U, V) \mid U \text{ open in } X \text{ and } V \text{ open in } Y\},$$

where $B(U, V)$ is the set of continuous maps $f\colon X \to Y$ such that $f(U) \subseteq V$ and any open cover of $f^{-1}(V)$ admits a finite subcover of U.

2.33 Fact: if X and Y are Hausdorff, this coincides with the more usual *compact-open topology*.

2.34 Theorem: if A is locally compact, then the rule $f \mapsto (x \mapsto (a \mapsto f(x, a)))$ defines a bijection

$$\mathrm{Top}(X \times A, Y) \;\cong\; \mathrm{Top}(X, \mathrm{Map}(A, Y))$$

for any topological spaces X and Y.

- Note that this is just a bijection between sets of continuous maps. We will discuss how this can be improved to a homeomorphism between mapping spaces in the next lecture.
- We can take $A = [0, 1]$, so homotopies may be equivalently viewed as maps to a pathspace. Using this, we can reformulate the definition of (Hurewicz) fibrations $p\colon E \to B$ as follows.

2.35 Reformulation: Let $\mathrm{Path}(E) = \mathrm{Map}([0, 1], E)$ with the open-open topology and let N_p be the construction of the proof of Proposition 2.26. Then p is a Hurewicz fibration if and only if the map $\mathrm{Path}(E) \to N_p$ defined by $\gamma \mapsto (\gamma(0), p \circ \gamma)$ admits a section.

- Such a section is called a *path-lifting function*. (If p is a covering space, then there exists a unique path-lifting function – in particular, covering spaces are Hurewicz fibrations.)
- Sketch of proof of part (b) of Theorem 2.30. Write $p|U$ for the restriction of p to $p^{-1}(U) \to U$ and $s_U\colon N_{p|U} \to \mathrm{Path}(p^{-1}(U))$ for a choice of path-lifting function. There is an open cover of N_p given by $V(U_1, \ldots, U_k)$ for $k \geqslant 1$ and $U_i \in \mathcal{U}$, where $(e, \gamma) \in V(U_1, \ldots, U_k)$ if and only if $\gamma([\frac{i-1}{k}, \frac{i}{k}]) \subseteq U_i$ for all i. Using $s_{U_1}, \ldots, s_{U_k}$ one can construct a path-lifting function for p defined on $V(U_1, \ldots, U_k)$. Using the fact that $\mathcal{U}$ is numerable, one can then "glue together" these various locally-defined path-lifting functions to obtain a globally-defined one using a partition of unity. The details of this gluing are quite intricate – see chapter 7 of [May] for the full details.

2.36 Definition: *fibre bundles*, *trivialising open covers* for (the base spaces of) fibre bundles.

2.37 Corollary:

(a) Every fibre bundle is a Serre fibration.
(b) Every fibre bundle that admits a *numerable* trivialising open cover is a Hurewicz fibration.

- Note: every open cover of a paracompact space has a numerable refinement, so every fibre bundle over a paracompact base space is a Hurewicz fibration.

7. Monday 29 October

2.38 Aside on a "convenient category of topological spaces".

(a) Definitions:
A topological space X is *compactly generated* if the following condition on a subset $A \subseteq X$ implies that it is closed in X:
($\bullet$) for every continuous $a\colon K \to X$ from a compact Hausdorff space, $a^{-1}(A)$ is closed in K.
A topological space X is *weakly Hausdorff* if, for every continuous map $a\colon K \to X$ from a compact Hausdorff space K, its image $a(K)$ is closed in X.

(b) Examples:
Any Hausdorff space is weakly Hausdorff.
All CW-complexes are compactly generated (and Hausdorff).
All locally compact spaces and all first-countable spaces (including metrisable spaces) are compactly generated.

(c) For any space X, let kX have the same underlying set, but modify its topology so that its closed subsets are precisely those satisfying condition ($\bullet$) above. Note that the identity is a continuous map $kX \to X$. Also note that this map is a weak equivalence.

(d) For any compactly generated space X, consider all equivalence relations on X with the property that the quotient is weakly Hausdorff. There is a *smallest* such equivalence relation: the corresponding quotient of X is denoted $X \to hX$.[2]

[2] Note: in contrast to $kX \to X$, the map $X \to hX$ is not always a weak equivalence, even if we assume that X is compactly generated to begin with. Let Z be a countably infinite set with the cofinite topology. This is first-countable, and therefore compactly generated. It is not path-connected (for a proof of this, see MO:48970). However, it is easy to see that hZ is the one-point space, so the map $Z \to hZ$ does not induce an injection on π_0, so it is not a weak equivalence.

(e) The operations k and h extend to functors in the following diagram:

$$\begin{tikzcd}
& \mathrm{WH} \arrow[dl, hook'] \arrow[dr, bend right, "k"'] & \\
\mathrm{Top} \arrow[dr, bend right, "k"'] & & \mathrm{CGWH} \arrow[ul, hook'] \arrow[dl, hook] \\
& \mathrm{CG} \arrow[ul, hook] \arrow[ur, bend left, "h"] &
\end{tikzcd} \tag{1}$$

In each case, k is a right adjoint to the inclusion and h is a left adjoint to the inclusion.

(f) Limits and colimits in CGWH: If we have a diagram in CGWH and take its colimit in Top, the resulting space will be in CG but not necessarily in WH. Since h is a left adjoint, it preserves colimits, so the colimit of the diagram in CGWH is obtained by applying the functor h to the colimit in Top:

$$\mathrm{colim}_{\mathrm{CGWH}} = h \circ \mathrm{colim}_{\mathrm{Top}}.$$

Similarly, if we have a diagram in CGWH and take its limit in Top, the resulting space will be in WH but not necessarily in CG. Since k is a right adjoint, it preserves limits, so the limit of the diagram in CGWH is obtained by applying the functor k to the limit in Top:

$$\lim_{\mathrm{CGWH}} = k \circ \lim_{\mathrm{Top}}.$$

(g) In many important cases, the functor h is unnecessary to compute colimits in CGWH. If we have a pushout diagram in CGWH in which one map is a closed inclusion, or if we have an infinite sequence of closed inclusions in CGWH, then the colimit (in Top) of either of these diagrams is already in CGWH, and we do not need to apply h. In particular this means that, to construct CW-complexes, we may use colimits either in CGWH or in Top – they are exactly the same thing for the relevant diagrams.

(h) Fact: if X and Y are (geometric realisations of) CW-complexes, then their product – in the category CGWH – is the geometric realisation of a CW-complex (built using cells corresponding to pairs of cells in X and in Y).

2.39 An important adjunction:

- For CGWH spaces X and Y, let $\mathrm{Map}(X,Y)$ be the set of continuous maps $X \to Y$ with the *k-ification* of the compact-open topology, i.e. the result of applying the operation k from 2.38(c) above to the compact-open topology. With this topology, $\mathrm{Map}(X,Y)$ is CGWH.
- Theorem: for all $X,Y,Z \in \mathrm{CGWH}$, there is a natural homeomorphism

$$\mathrm{Map}(X \times Y, Z) \;\cong\; \mathrm{Map}(X, \mathrm{Map}(Y,Z)).$$

- If X and Y are based CGWH spaces, let $\mathrm{Map}_*(X,Y) \subseteq \mathrm{Map}(X,Y)$ be the set of continuous maps that preserve the basepoint, with the subspace topology. Let $X \wedge Y$ be the smash product $(X \times Y)/(X \vee Y)$ where $X \vee Y = (X \times \{*\}) \cup (\{*\} \times Y)$. Here we of course take the CGWH product topology on $X \times Y$, i.e. the k-ification of the usual product topology. The quotient $X \wedge Y$ is again in CGWH because of the following general fact.
- If Z is a CGWH space and $\sim$ is an equivalence relation on Z such that $\{(z,\bar{z}) \mid z \sim \bar{z}\} \subseteq Z \times Z$ is closed, then $Z/{\sim}$ is also CGWH.[3]
- Theorem: for all based CGWH spaces X,Y,Z, there is a natural homeomorphism

$$\mathrm{Map}_*(X \wedge Y, Z) \;\cong\; \mathrm{Map}_*(X, \mathrm{Map}_*(Y,Z)).$$

- Corollary: taking $Y = S^1$ and applying π_0, we see that for based CGWH spaces X, Z there is a natural bijection

$$\langle \Sigma X, Z\rangle \;\cong\; \langle X, \Omega Z\rangle,$$

just as in Proposition 1.21.

($*$) Now we return to fibrations and cofibrations, and go back to working in the category Top of all topological spaces (for the moment).

[3] Without this condition, it is CG but not necessarily WH. Taking the quotient and applying h to the result is equivalent to taking the closure of the equivalence relation in $Z \times Z$ and then taking the quotient.

2.40 Lemma: Suppose that f is a retract of g in the category Map of continuous maps between spaces. If g is a (cofibration/fibration/homotopy equivalence), then so is f.
Proof: easy diagram chase.

2.41 Theorem: a strengthening of some things that we proved earlier about (co)fibrations:
(a) Any map $f\colon X \to Y$ may be factorised as

$$\begin{tikzcd} & Z \arrow[dr] & \\ X \arrow[ur, hook] \arrow[rr, "f"] \arrow[dr, hook] & & Y, \\ & Z' \arrow[ur] & \end{tikzcd} \tag{2}$$

where the left-hand maps are closed cofibrations and the right-hand maps are fibrations, and the maps $X \to Z'$ and $Z \to Y$ are homotopy equivalences.
[This strengthens (2.11) and (2.26).]
(b) Consider the commutative square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & B, \end{tikzcd} \tag{3}$$

where i is a closed cofibration and p is a fibration. If either i or p is a homotopy equivalence, then there is a map $X \to E$ making the two triangles commute.

- Note that, for any space Y, the inclusion $Y \times \{0\} \hookrightarrow Y \times [0,1]$ is a closed cofibration and the evaluation map $\mathrm{ev}_0\colon \mathrm{Path}(Y) \to Y$ is a fibration, and these are both homotopy equivalences. The statement of (b) therefore includes as special cases the statements that closed cofibrations have the HEP for all spaces Y and fibrations have the HLP for all spaces Y.

2.42 Definition: a *model category* is a category $\mathcal{C}$ with all (small) limits and colimits, together with a choice of three subclasses of morphisms of $\mathcal{C}$, called Weq, Cof and Fib, such that:
(1) Weq contains all isomorphisms and, for composable morphisms f and g, if two out of $\{f, g, gf\}$ are in Weq, then so is the third.
(2) In the diagram (3), if $i \in$ Cof and $p \in$ Fib, and either $i \in$ Weq or $p \in$ Weq, then there is a morphism $X \to E$ making the two triangles commute.
(3) Each morphism $f\colon X \to Y$ of $\mathcal{C}$ may be factorised as

$$\begin{tikzcd} & Z \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "f"] \arrow[dr, "i'"'] & & Y, \\ & Z' \arrow[ur, "p'"'] & \end{tikzcd} \tag{4}$$

with $i \in \mathrm{Cof}$, $p \in \mathrm{Fib} \cap \mathrm{Weq}$, $i' \in \mathrm{Cof} \cap \mathrm{Weq}$ and $p' \in \mathrm{Fib}$.
(4) The three classes Cof, Fib, Weq are closed under taking retracts of morphisms in $\mathcal{C}$.

2.43 Remark: by Lemma 2.40 and Theorem 2.41, the category Top has a structure of a model category given by
- Weq = {homotopy equivalences}
- Fib = {Hurewicz fibrations}
- Cof = {closed cofibrations}

2.44 Note: an equivalent definition of model category is given by axioms (1), (3) and:
(2)′ In the diagram (3),

$$\begin{aligned} i \in \mathrm{Cof} &\iff \text{for any } p \in \mathrm{Fib} \cap \mathrm{Weq} \ \ldots \\ i \in \mathrm{Cof} \cap \mathrm{Weq} &\iff \text{for any } p \in \mathrm{Fib} \ \ldots \\ p \in \mathrm{Fib} &\iff \text{for any } i \in \mathrm{Cof} \cap \mathrm{Weq} \ \ldots \\ p \in \mathrm{Fib} \cap \mathrm{Weq} &\iff \text{for any } i \in \mathrm{Cof} \ \ldots \end{aligned}$$

... for any morphisms f, g of $\mathcal{C}$ making the square commute, there exists a diagonal morphism $h\colon X \to E$ making the two triangles commute.

[Note that axiom (2) is exactly the implications $\Longrightarrow$.]

- It then follows that $(\mathrm{Weq}, \mathrm{Fib})$ determines Cof and that $(\mathrm{Weq}, \mathrm{Cof})$ determines Fib.

2.45 We may use this language to concisely summarise the key properties of several variations on the notion of fibration and cofibration. There exist model structures on Top with:

	Weq	Fib	Cof
(H)	homotopy equivalences	Hurewicz fibrations	closed cofibrations
(Q)	weak homotopy equivalences	Serre fibrations	(1)
(M)	weak homotopy equivalences	Hurewicz fibrations	(2)

The class (1) is smaller than the class of cofibrations, but contains all CW-pairs. Specifically, it is the smallest class of maps that contains $S^{n-1} \hookrightarrow D^n$ for all $n \geqslant 0$ and is closed under retracts, disjoint union, pushouts and (possibly infinite) compositions.[4] The class (2) is necessarily larger than the class (1) (fewer fibrations with the same weak equivalences implies more cofibrations), so in particular it also contains all CW-pairs.

- Moreover, we also obtain three different model structures on each of the other categories in the diagram

$$\begin{tikzcd} \mathrm{CGWH}_* \arrow[r] \arrow[d, hook] & \mathrm{CGWH} \arrow[d, hook] \\ \mathrm{Top}_* \arrow[r] & \mathrm{Top}, \end{tikzcd}$$

where the vertical functors are inclusions of full subcategories and the horizontal functors forget the basepoint of a space, by defining $\mathrm{Weq}_{(\mathrm{H})} \subseteq \mathrm{Mor}(\mathcal{C})$ to be the class of all morphisms that are mapped into $\mathrm{Weq}_{(\mathrm{H})} \subseteq \mathrm{Mor}(\mathrm{Top})$, etc., for each $\mathcal{C}$.

8. Wednesday 31 October

2.46 Two remarks:

(a) In the category CGWH, if $f\colon X \to Y$ is a Serre fibration between CW-complexes, then it is a Hurwicz fibration.

- Note that, since all CW-complexes are objects of CGWH and a map in CGWH is a Serre fibration if and only if it is a Serre fibration in Top, an equivalent way to state fact (a) is the following: if $f\colon X \to Y$ is a Serre fibration between CW-complexes, then it has the homotopy lifting property for all CGWH spaces.

(b) In Top_* (or CGWH_*) we may define a notion of *based (co)fibration* by using exactly the same lifting/extension diagrams, but taking all spaces and maps to be based. However, in the model structures discussed in (2.45) above, we use the *unbased* versions of these notions: for example, the fibrations in the (H) model structure on Top_* are exactly those based maps that are Hurewicz fibrations when we ignore basepoints.

- Fortunately the notions of based and unbased (co)fibrations are not too different, as we'll see in a moment.

2.47 Definition: a based space X is *non-degenerately based*, or *well-based*, if the inclusion of the basepoint $\{*\} \hookrightarrow X$ is a cofibration.

- Note that every based space is homotopy equivalent (although not necessarily *based* homotopy equivalent) to a well-based space: just factor the inclusion $i\colon \{*\} \hookrightarrow X$ through the mapping cylinder of i:

$$\{*\} \hookrightarrow M_i \longrightarrow X.$$

In other words $X \simeq X \vee [0,1]$, where we use the basepoint 0 of the interval to take the wedge, but then consider the wedge sum $X \vee [0,1] = M_i$ as a based space with the basepoint 1.

2.48 Proposition: Let $f\colon X \to Y$ be a map of based spaces.

(i) If f is an unbased cofibration, then f is a based cofibration.
The reverse implication also holds if X and Y are well-based.

[4] For a diagram in Top indexed by an ordinal, the transfinite composition is defined recursively, in the usual way for a successor ordinal and by taking the colimit for a limit ordinal.

(ii) If f is a based fibration, then f is an unbased fibration.
If f is an unbased fibration, then f has the based HLP for all well-based spaces.

(iii) For Serre fibrations, the statement is simpler, since all cubes are well-based:
The map f is a based Serre fibration if and only if it is an unbased Serre fibration.

($*$) From now on, we will work in the full subcategory of CGWH spaces, unless otherwise specified (although many things will also work in the category of all topological spaces).

- In particular, limits and colimits will be taken in this category. For limits, this means computing the limit in Top and then (if necessary) applying the functor k. This is only a mild complication, since kX is always weakly homotopy equivalent to X. For colimits, we have to compute the colimit in Top and then (if necessary – i.e. if the result is not already weakly Hausdorff) apply the functor h. This is a bigger problem than in the case of limits, because there are compactly generated spaces Z such that hZ is not weakly homotopy equivalent to Z (see exercise D of the optional exercises). However, for the kinds of diagrams that we will need to take colimits of (sequences of closed inclusions, pushout diagrams in which at least one map is a closed inclusion), the colimit in Top will already be weakly Hausdorff, so this issue will not arise.
- The mapping space $\mathrm{Map}(X, Y)$ will always be given the k-ification of the compact-open topology.
- Quotients: for any equivalence relation $\sim$ on a CGWH space X, the quotient $X/{\sim}$ is always CG, and it is WH if and only if $\sim$ is closed as a subspace of $X \times X$. We may therefore obtain a CGWH space in two different ways: take the closure of the equivalence relation in $X \times X$ and then take the quotient, or take the quotient by $\sim$ and then apply the functor h; these are naturally homeomorphic: $X/\overline{\sim} \cong h(X/{\sim})$. We will always implicitly do this when taking quotients in CGWH (but we will very rarely need non-closed equivalence relations, so we will almost always just take the ordinary quotient).

2.49 Constructions:

(a) The *double mapping cylinder* $M(X, f, g)$ for maps $f : Y \leftarrow X \to Z : g$ in Top_*.

(b) The *double pathspace* $P(X, f, g)$ for maps $f : Y \to X \leftarrow Z : g$ in Top_*.

- If f and g are inclusions of subspaces, we write $P(X, Y, Z) = P(X, f, g)$.
- Some special cases are:

$$\begin{aligned}
CX &= M(X, \mathrm{id}_X, X \to \{*\}) \\
\Sigma X &= M(X, X \to \{*\}, X \to \{*\}) \\
M_f &= M(X, f, \mathrm{id}_X) \qquad \text{(mapping cylinder)} \\
C_f &= M(X, f, X \to \{*\}) \quad \text{(homotopy cofibre)} \\
PX &= P(X, X, \{*\}) \\
\Omega X &= P(X, \{*\}, \{*\}) \\
N_f &= P(X, f, X) \\
F_f &= P(X, f, \{*\}) \qquad \text{(homotopy fibre)}
\end{aligned}$$

2.50 Definition: the (based) *cofibre sequence* of a map $f \colon X \to Y$ in Top_*:

$$X \xrightarrow{f} Y \xrightarrow{i(f)} C_f \xrightarrow{\pi(f)} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i(f)} \Sigma C_f \xrightarrow{-\Sigma\pi(f)} \Sigma^2 X \xrightarrow[\simeq (-\Sigma)^2 f]{\Sigma^2 f} \Sigma^2 Y \longrightarrow \cdots$$

2.51 Theorem: For any $Z \in \mathrm{Top}_*$, applying the functor $\langle -, Z\rangle \colon \mathrm{Top}_* \to \mathrm{Set}_*$ to this sequence results in an exact sequence of pointed sets.

2.52 If we apply $\langle -, Z\rangle$ to the cofibre sequence, we get a sequence of groups starting from the 4th term, and a sequence of abelian groups from the 7th term. A sequence of (abelian) groups is exact if and only if it is exact when regarded as a sequence of pointed sets, so Theorem 2.51 tells us that cofibre sequences give rise to exact sequences of (abelian) groups from the 4th term (resp. 7th term) term onwards.

2.53 Definition: let us call a diagram $A \to B \to C$ in Top_* a *triple*.

- A triple is called *coexact* if the induced sequence $\langle A, Z\rangle \leftarrow \langle B, Z\rangle \leftarrow \langle C, Z\rangle$ is exact for all $Z \in \mathrm{Top}_*$.

- More generally, we say that a sequence in Top_* is *coexact* if it is taken to an exact sequence in Set_* by $\langle -, Z\rangle$ for every $Z \in \mathrm{Top}_*$. So Theorem 2.51 says that cofibre sequences are coexact.
- Two triples $A \to B \to C$ and $A' \to B' \to C'$ are *homotopy equivalent* if there are homotopy equivalences $A \to A'$, $B \to B'$ and $C \to C'$ making the squares commute up to homotopy. (Here we mean *based* homotopy equivalences and "up to *based* homotopy".)
- Observation: *homotopy equivalences of triples preserve coexactness.*

2.54 Lemma:

(a) Triples of the form
$$X \xrightarrow{f} Y \xrightarrow{i(f)} C_f \tag{5}$$
are coexact.

(b) Each triple of the form
$$Y \xrightarrow{i(f)} C_f \xrightarrow{\pi(f)} \Sigma X \tag{6}$$
is homotopy equivalent to a triple of the form (5).

(c) Each triple of the form
$$C_f \xrightarrow{\pi(f)} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \tag{7}$$
is homotopy equivalent to a triple of the form (6).

(d) Applying the functor $-\Sigma\colon \mathrm{Top}_* \to \mathrm{Top}_*$ to a triple of the form (5)/(6)/(7) results in a triple that is homotopy equivalent to one of the form (5)/(6)/(7).

- Note that Theorem 2.51 follows from this sequence of lemmas.
- Proof (given in detail in the lectures, but just a sketch here):

(a) This is easy to see, unwinding the definitions.

(d) This is also easy to construct. For example, for $-\Sigma$ of a triple of the form (5), we can construct a homotopy equivalence of triples to one of the form (5) (with f replaced by $-\Sigma f$) using the homeomorphism $\Sigma C_f \cong C_{-\Sigma f}$ that swaps the cone and suspension coordinates and inverts the suspension coordinate.

(b+c) This is a little more tricky. There is a map $\varphi\colon C_{i(f)} \to \Sigma X$ collapsing the subspace $CY \subseteq C_{i(f)}$ to a point, which makes the appropriate squares commute up to homotopy. Using the fact that $i(f)\colon Y \hookrightarrow C_f$ is a cofibration, we may construct a homotopy inverse for φ as
$$\Sigma X \twoheadleftarrow C_f \xrightarrow{x \mapsto (x,1)} C_f \times [0,1] \xrightarrow{r} M_{i(f)} \twoheadrightarrow C_{i(f)},$$
where the wrong-way arrow on the left means that the map $C_f \to C_{i(f)}$ factors through the quotient $C_f \twoheadrightarrow \Sigma X$ and therefore induces a well-defined map $\Sigma X \to C_{i(f)}$. Therefore (6)/(7) are homotopy equivalent to (5)/(6) with f replaced by $i(f)$.

9. Monday 5 November

2.55 Definition: the (based) *fibre sequence* of a map $f\colon X \to Y$ in Top_*:
$$\cdots \longrightarrow \Omega^2 X \xrightarrow[\simeq (-\Omega)^2 f]{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega j(f)} \Omega F_f \xrightarrow{-\Omega p(f)} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{j(f)} F_f \xrightarrow{p(f)} X \xrightarrow{f} Y$$

2.56 Definition: a sequence in Top_* is *exact* if it is taken to an exact sequence in Set_* by the contravariant functor $\langle Z, -\rangle$ for every $Z \in \mathrm{Top}_*$.

2.57 Theorem: the fibre sequence of any map $f\colon X \to Y$ in Top_* is exact.

- Proof: dual to that of Theorem 2.51 (last week).

2.58 Lemma: there are natural isomorphisms $\pi_n(P(X, A, \{*\})) \cong \pi_{n+1}(X, A, *)$ for any pair of spaces (X, A) and all $n \geqslant 0$.

- Note that $P(X, A, \{*\}) = F_i$, where i is the inclusion $A \hookrightarrow X$.

2.59 Corollary: For any pair of spaces (X, A), applying the functor $\pi_0 = \langle S^0, - \rangle \colon \mathrm{Top}_*^{\mathrm{op}} \to \mathrm{Set}_*$ to the fibre sequence of the inclusion $A \hookrightarrow X$ recovers the long exact sequence of the pair (X, A) (this was Proposition 1.9).

2.60 Definition: a map $f \colon X \to Y$ in Top is a *quasifibration* if and only if the inclusion

$$f^{-1}(y) \hookrightarrow F_{f,y}$$

is a weak homotopy equivalence for each $y \in Y$. (Note: since Y is not a based space, we have to choose a basepoint to define the homotopy fibre F_f, and the notation $F_{f,y}$ means the homotopy fibre of f using $y \in Y$ as the basepoint.)

2.61 Lemma: if $f \colon X \to Y$ in Top_* is a quasifibration, then there is a long exact sequence of (pointed sets/groups) the form

$$\cdots \to \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \to \pi_{n-1}(F) \to \pi_{n-1}(X) \xrightarrow{f_*} \pi_{n-1}(Y) \to \cdots,$$

where $F = f^{-1}(*)$.

- Proof: apply π_0 to the fibre sequence of f and replace F_f with $f^{-1}(*)$ via the isomorphisms on π_n induced by the inclusion.

2.62 Proposition: The following implications hold for a map $f \colon X \to Y$ in Top.

$$\begin{array}{ccc}
\boxed{f \text{ is a Hurewicz fibration}} & \Rightarrow & \boxed{f \text{ is a Serre fibration}} \\
\Downarrow & & \Downarrow \\
\boxed{\begin{array}{c} f^{-1}(y) \hookrightarrow F_{f,y} \text{ is a homotopy} \\ \text{equivalence for all } y \in Y \end{array}} & \Rightarrow & \boxed{f \text{ is a quasifibration}}
\end{array}$$

- Proof (given in detail in the lectures, but just a sketch here):
  · The horizontal implications are immediate from the definitions.
  · The left-hand vertical implication uses the viewpoint of Hurewicz fibrations as maps for which there is a section (path-lifting function) of the map

$$\mathrm{Path}(X) = P(X, X, X) \longrightarrow P(Y, f, Y) = N_f$$

  in order to construct homotopy inverses for the inclusions $f^{-1}(y) \hookrightarrow F_{f,y}$.
  · For the right-hand vertical implication: Let $y \in Y$ and write $F = f^{-1}(y)$. We first showed that the map $f \colon (X, F) \to (Y, \{y\})$ induces isomorphisms on all relative homotopy groups if f is a Serre fibration. There is then a map of exact sequences from the LES of the pair (X, F) to π_0 of the fibre sequence of f. One third of these maps are the identities $\pi_n(X) = \pi_n(X)$. Another third are the maps $f_* \colon \pi_n(X, F) \to \pi_n(Y)$ that we just showed are isomorphisms. The last third are the maps $\pi_n(F) \to \pi_n(F_{f,y})$ induced by the inclusion $F \hookrightarrow F_{f,y}$. By the 5-lemma (plus a small extra argument for small n, where the sequences are of just pointed sets or groups, rather than abelian groups), these are also isomorphisms, so f is a quasifibration.

2.63 Lemma: if f is a quasifibration in Top_*, its induced long exact sequence (from Lemma 2.61) is isomorphic to the LES of the pair $(X, f^{-1}(*))$.

- Proof: this follows from the map of long exact sequences considered in the last part of the proof above.

2.64 Proposition: the cofibre and fibre sequences of morphisms in Top_* are *natural*, in other words a commutative square of maps induces a map of sequences in Top_*. In particular, this implies that the LES on π_* induced by a quasifibration is also natural.

- Proof.
- Some important examples of fibre bundles (and therefore quasifibrations, inducing long exact sequences of homotopy groups) are the *Hopf bundles*.
- Let $\mathbb{F}$ be one of the four $\mathbb{R}$-algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (real numbers, complex numbers, quaternions, octonions) and let $n \geqslant 2$ be an integer. (If $\mathbb{F} = \mathbb{O}$ we assume that $n = 2$ or 3.) There is a

continuous action of the topological group $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ on $\mathbb{F}^n$ given by multiplication in each coordinate.[5] Restricted to $\mathbb{F}^n \setminus \{0\}$, this is a *free* action, whose quotient is called the *nth projective space* $\mathbb{FP}^{n-1}$.

2.65 Proposition: the quotient map $q\colon \mathbb{F}^n \setminus \{0\} \longrightarrow \mathbb{FP}^{n-1}$ is a fibre bundle (and therefore also a Hurewicz fibration, since the base space is compact, so paracompact).

- We therefore have a fibre sequence (up to homotopy equivalence) of the form

$$\mathbb{F}^\times \longrightarrow \mathbb{F}^n \setminus \{0\} \longrightarrow \mathbb{FP}^{n-1}.$$

- Note: We usually just write the first three terms of a fibre sequence, and it is understood that it continues infinitely to the left. In any case, a fibre sequence is determined by the first map (by definition!), so we are not losing information this way.
- Topologically, the first two spaces are just Euclidean spaces minus the origin, so they are homotopy equivalent to spheres and we may rewrite this homotopy fibre sequence as

$$S^{d-1} \longrightarrow S^{nd-1} \longrightarrow \mathbb{FP}^{n-1}$$

for $d = 1, 2, 4, 8$.

2.66 Example: In particular, setting $n = 2$ we obtain four homotopy fibre sequences

$$\begin{aligned}
&(\mathbb{R}) && S^0 \longrightarrow S^1 \xrightarrow{2} S^1 = \mathbb{RP}^1\\
&(\mathbb{C}) && S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2 = \mathbb{CP}^1\\
&(\mathbb{H}) && S^3 \longrightarrow S^7 \xrightarrow{\nu} S^4 = \mathbb{HP}^1\\
&(\mathbb{O}) && S^7 \longrightarrow S^{15} \xrightarrow{\sigma} S^8 = \mathbb{OP}^1
\end{aligned}$$

In the first case, we just obtain the double covering of S^1 over itself. In the second case, the map $\eta\colon S^3 \to S^2$ is the famous *Hopf bundle* (more generally the maps $S^{nd-1} \to \mathbb{FP}^{n-1}$ are often called Hopf bundles too). Using the long exact sequence of homotopy groups induced by the fibre sequence of η, and the fact that the higher homotopy groups of S^1 vanish (because its universal cover is $\mathbb{R}$, which is contractible), we see that η induces isomorphisms

$$\pi_i(S^3) \;\cong\; \pi_i(S^2)$$

for all $i \geqslant 3$.

3. The Blakers-Massey theorem

3.1 Theorem: Let X be a space and $A, B \subseteq X$ open subsets such that $X = A \cup B$ and $A \cap B \neq \varnothing$. We summarise this by saying that (X, A, B) is a *triad*[6] Assume that the inclusions

$$\begin{aligned}
&A \cap B \hookrightarrow A\\
\text{and}\quad &A \cap B \hookrightarrow B
\end{aligned}$$

are p-connected and q-connected respectively, for integers $p, q \geqslant 0$. (We summarise this by saying that the triad (X, A, B) is (p, q)*-connected.*) Then the inclusion

$$P(A, A, A \cap B) \hookrightarrow P(X, A, B)$$

is $(p + q - 1)$-connected.

[5] The construction that we give here works as long as $\mathbb{F}$ is an associative $\mathbb{R}$-algebra, which is the case for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. However, $\mathbb{O}$ is non-associative, and therefore the construction that we give does not work. The problem is that $\mathbb{O}^\times$ is not a group, so we cannot produce an equivalence relation on $\mathbb{O}^n \setminus \{0\}$ using an action of $\mathbb{O}^\times$. One can get around this by trying to define an appropriate equivalence relation on $\mathbb{O}^n \setminus \{0\}$ directly, to obtain $\mathbb{OP}^{n-1}$. This can be done for $n = 2, 3$ with some care, using the fact that $\mathbb{O}$ is 2-associative (any subalgebra of it that is generated by 2 elements is associative). This is why we have to assume that $n = 2$ or 3 when $\mathbb{F} = \mathbb{O}$; there is no object called $\mathbb{OP}^k$ for $k \geqslant 3$. One could try to continue: the *Cayley-Dickson construction* D takes an algebra with involution to another algebra with involution. If we start with $\mathbb{R}$ with the trivial involution, then the algebras $\mathbb{C}, \mathbb{H}, \mathbb{O}$ are precisely $D^r(\mathbb{R})$ for $r = 1, 2, 3$. So we could try to produce projective spaces from $D(\mathbb{O}) = D^4(\mathbb{R})$, the *sedenions*, and more generally $D^r(\mathbb{R})$ for all r. However, $D^r(\mathbb{R})$ is not associative — nor even 2-associative — for $r \geqslant 4$, so we cannot apply the trick used in the case of $\mathbb{O}$ to define a projective plane.

[6] This is often called an *excisive triad*.

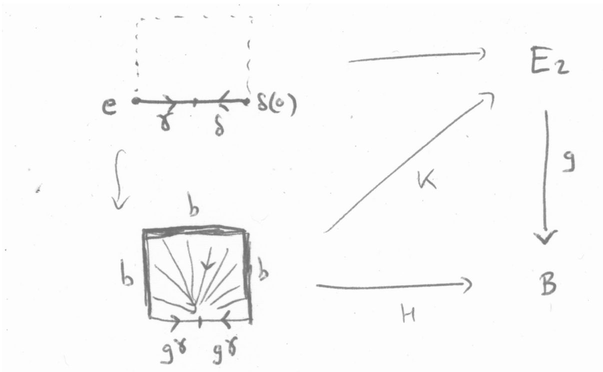
- Proof: next lecture(s).

3.2 Lemma: Let $i\colon E_1 \hookrightarrow E_2$ be an inclusion and let $g\colon E_2 \to B$ be a Serre fibration such that $f = g \circ i\colon E_1 \to B$ is also a Serre fibration (this is sometimes called a *relative Serre fibration*):

$$\begin{tikzcd} E_1 \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & E_2 \arrow[dl, "g"] \\ & B & \end{tikzcd}$$

Also fix $n \geqslant 0$. Then $i\colon E_1 \hookrightarrow E_2$ is n-connected if and only if for every $b \in B$, the restriction of i to $i_b\colon f^{-1}(b) \hookrightarrow g^{-1}(b)$ is n-connected.

- Proof: for $b \in B$ there is a map of fibre sequences, and therefore a map of long exact sequences. If we can prove the lemma for $n = 0, 1$, then for $n \geqslant 2$ we will be able to prove it just using the 5-lemma (or, more precisely, the two 4-lemmas), since we have a map of long exact sequences of *abelian groups* and we can do homological algebra. (Note: the fact that we have to say "for all $b \in B$" in the statement comes from the fact that "n-connected" for a map means that it induces isomorphisms/surjections on homotopy groups in a certain range of degrees *for all possible choices of basepoint.*) It therefore remains to prove the lemma for $n = 0$ and $n = 1$.
- Proof for $n = 0$: the direction $\Leftarrow$ is immediate from the definitions. For the direction $\Rightarrow$: we know that any point $e \in E_2$ is connected by a path γ to a point in E_1, and we need to show that there is moreover a path contained within the fibre $g^{-1}(g(e))$ that connects e to a point of $E_1 \cap g^{-1}(g(e)) = f^{-1}(g(e))$. We can do this by first lifting the path $g \circ \gamma$ up the Serre fibration f to a path δ that also covers $g \circ \gamma$ and $\delta(1) = \gamma(1)$. We then use the HLP for g as follows:



The restriction of K to the left, top and right sides of the square is a path contained in the fibre $g^{-1}(b)$ where $b = g(e)$ starting at e and ending at $\delta(0) \in f^{-1}(b)$.
- Proof for $n = 1$. Let $b \in B$ and $e \in f^{-1}(b)$. We'll show that the map induced by inclusion

$$\pi_1(g^{-1}(b), f^{-1}(b), e) \longrightarrow \pi_1(E_2, E_1, e) \tag{8}$$

is an isomorphism. The proof of surjectivity is quite similar to the second half of the proof for $n = 0$ above. Let γ be a path starting at e and ending in E_1, representing an element $[\gamma] \in \pi_1(E_2, E_1, e)$. As above, lift $g \circ \gamma$ up f to a path δ. Note that $[\gamma] = [\gamma * \bar{\delta}] \in \pi_1(E_2, E_1, e)$, where $\bar{\delta}$ means the reverse of δ. Using the HLP for g exactly as above and restricting K to the left, top and right sides of the square, we obtain a path ν such that $[\nu] = [\gamma * \bar{\delta}]$. Clearly $[\nu]$ is in the image of the map (8).

· For injectivity, see Figure 1 on page 49. The restriction of L to the back face of the 3-cube is a homotopy $\gamma * e \simeq \delta * e$ through maps of triples $([0,1], \{0\}, \{1\}) \to (g^{-1}(b), f^{-1}(b), \{e\})$, and therefore

$$[\gamma] = [\gamma * e] = [\delta * e] = [\delta] \in \pi_1(g^{-1}(b), f^{-1}(b), e).$$

3.3 Corollary: (*version II of the Blakers-Massey theorem*) If (X, A, B) is a (p, q)-connected triad for $p, q \geqslant 0$, then the map induced by inclusion

$$\pi_n(A, A \cap B) \longrightarrow \pi_n(X, B)$$

is an isomorphism for $n \leqslant p+q-1$ and surjective for $n \leqslant p+q$.

- Note: this is analogous to the excision theorem for homology, where π_n is replaced with H_n. For homology, there is no hypothesis on the connectivity of the triad, and the map induced by the inclusion $(A, A\cap B) \hookrightarrow (X, B)$ is an isomorphism on relative homology in *all* degrees. The Blakers-Massey theorem is therefore a weak form of excision for homotopy groups.
- Proof of Corollary 3.3: Evaluation of a path at 0 defines a relative Hurewicz fibration (and therefore a relative Serre fibration)

$$\begin{tikzcd} P(A,A,A\cap B) \arrow[rr, "i"] \arrow[dr, "\mathrm{ev}_0"'] & & P(X,A,B) \arrow[dl, "\mathrm{ev}_0"] \\ & A & \end{tikzcd}$$

Apply Lemma 3.2 to this, and recall that in general $\pi_n(P(Z,Y,\{*\})) \cong \pi_{n+1}(Z,Y,*)$ (see Lemma 2.58).

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3.4 Corollary: (*version III of the Blakers-Massey theorem*) Let (X,A,B) be a triad and assume:

$$\pi_n(A, A\cap B) = 0 \text{ for } 1 \leqslant n \leqslant p$$
$$\pi_n(B, A\cap B) = 0 \text{ for } 1 \leqslant n \leqslant q,$$

for integers $p, q \geqslant 0$. Then the map induced by inclusion

$$\pi_n(A, A\cap B) \longrightarrow \pi_n(X,B)$$

is an isomorphism for $1 \leqslant n \leqslant p+q-1$ and surjective for $1 \leqslant n \leqslant p+q$.

- Note: this is almost identical to version II (Corollary 3.3), except that there is no hypothesis on $\pi_0(A, A\cap B)$ or $\pi_0(B, A\cap B)$, and no conclusion about the induced map on π_0.
- Proof, by applying Corollary 3.3 to a related triad, with extra *ad hoc* arguments to deal with π_0 issues.

3.5 Corollary: (*version IV of the Blakers-Massey theorem*) Let (X,A,B) be a triad. If $(A, A\cap B)$ is p-connected then (X,B) is also p-connected.

- Proof: Apply Corollary 3.4 with $q=0$. This tells us that $\pi_n(X,B) = 0$ for $1 \leqslant n \leqslant p$. Then an easy extra argument shows that $\pi_0(X,B) = 0$ too.

3.6 Definition of the *suspension homomorphism* Σ_* for a space $X \in \mathrm{Top}_*$ and degree $i \geqslant 0$:

$$\pi_i(X,*) \xleftarrow{\cong} \pi_{i+1}(CX, X, *) \longrightarrow \pi_{i+1}(\Sigma X, \{*\}, *) = \pi_{i+1}(\Sigma X, *),$$

where the first arrow comes from the LES for the pair (CX, X) and is an isomorphism because CX is contractible, and the second arrow is induced by the quotient map $q\colon CX \to \Sigma X$ that collapses the base of the cone to a point.

3.7 Lemma: there is a commutative diagram

$$\begin{tikzcd} \pi_i(X,*) \arrow[rr, bend left, "\Sigma_*"] & \pi_{i+1}(C_+X, X, *) \arrow[l, "\cong"'] \arrow[r, "q_*"] \arrow[d, "\mathrm{incl}_*"'] & \pi_{i+1}(\Sigma X, *) \arrow[dl, "\cong"', "\mathrm{incl}_*"] \\ & \pi_{i+1}(\Sigma X, C_-X, *), & \end{tikzcd}$$

where C_+X and C_-X are the upper and lower cones of the suspension ΣX respectively, and the diagonal map is an isomorphism since it is part of the LES for the pair $(\Sigma X, C_-X)$, and C_-X is contractible.

3.8 Theorem: (Freudenthal) If $X \in \mathrm{Top}_*$ is n-connected, then

$$\Sigma_* \colon \pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$$

is an isomorphism for $i \leqslant 2n$ and surjective for $i \leqslant 2n+1$.

- Proof: we will apply the Blakers-Massey theorem (version II, i.e. Corollary 3.3) to the triad $(\Sigma X, C_+X, C_-X)$. To be rigorous we should thicken C_+X and C_-X slightly to open cones on X contained in ΣX in order for this to be a triad, but this does not change the homotopy types of any of the spaces, pairs of spaces or maps involved. Since X is n-connected, the LES of the pair (CX, X) implies that (CX, X) is $(n+1)$-connected, so the triad $(\Sigma X, C_+X, C_-X)$ is $(n+1, n+1)$-connected. Corollary 3.3 then tells us that the vertical map in the diagram above is an isomorphism for $i+1 \leqslant 2n+1$ and surjective for $i+1 \leqslant 2n+2$.

3.9 Example: for any $i < n$, any map $S^i \to S^n$ is nullhomotopic (by the cellular approximation theorem), so S^n is $(n-1)$-connected. Theorem 3.8 therefore implies that

$$\Sigma_* \colon \pi_i(S^n) \longrightarrow \pi_{i+1}(S^{n+1})$$

is an isomorphism for $i \leqslant 2n-2$ and surjective for $i \leqslant 2n-1$. In particular, we have

$$\pi_1(S^1) \twoheadrightarrow \pi_2(S^2) \xrightarrow{\cong} \pi_3(S^3) \xrightarrow{\cong} \pi_4(S^4) \xrightarrow{\cong} \cdots$$

We know from covering space theory that $\pi_1(S^1) \cong \mathbb{Z}$, and then from the LES associated to the Hopf fibre sequence $S^1 \to S^3 \to S^2$ that $\pi_2(S^2) \cong \mathbb{Z}$. Any surjection $\mathbb{Z} \to \mathbb{Z}$ must be an isomorphism, so the first map above must also be an isomorphism. In particular this gives a proof that $\pi_n(S^n) \cong \mathbb{Z}$ for all $n \geqslant 1$ (this can also be proved using the Hurewicz theorem).

3.10 We can put the abelian groups $\pi_i(S^n)$ into a table with n on the horizontal axis and $j = i-n$ on the vertical axis. The suspension homomorphisms are then arrows that go one step to the right. They are isomorphisms underneath the line $j = n-2$, and surjections just above it. We have $\mathbb{Z}/2$ at the origin (since $\pi_0(S^0) \cong \mathbb{Z}/2$) and then $\mathbb{Z}$ on the rest of the x-axis. For each fixed j, the Freudenthal suspension theorem implies that the entries on the jth row *stabilise* after $\pi_{2j+2}(S^{j+2})$. The stable value is called π_j^s (the *jth stable homotopy group of the spheres*), equivalently, we can just define $\pi_j^s = \pi_{2j+2}(S^{j+2})$. By the previous discussion, we know that $\pi_0^s \cong \mathbb{Z}$. From the Hopf fibre sequence we also know that $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$, which is just above the stable range, so we have a surjection $\mathbb{Z} \twoheadrightarrow \pi_4(S^3) \cong \pi_1^s$. It turns out in fact that $\pi_1^s \cong \mathbb{Z}/2$. Later in the course, we'll prove that almost all of the homotopy groups of spheres are finite, and in particular that π_j^s is finite for all $j \geqslant 1$.

- Now we'll begin the proof of the Blakers-Massey theorem (Theorem 3.1).

3.11 Notation: a *cube* will mean any space of the form $\prod_{i=1}^n [a_i, b_i]$ for $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ such that, for some $\ell > 0$ and for every $i \in \{1, \ldots, n\}$, either $b_i = a_i$ or $b_i = a_i + \ell$. The *edge length* of this cube is ℓ. A *face* of this cube is any cube $\prod_{i=1}^n [a_i', b_i']$ such that, for each $i \in \{1, \ldots, n\}$, the interval $[a_i', b_i']$ is either $[a_i, b_i]$ or $\{a_i\}$ or $\{b_i\}$. If $t \in C = \prod_{i=1}^n [a_i, b_i]$, its ith coordinate t_i is called *small* if $t_i < \frac{1}{2}(b_i + a_i)$. Note that this is impossible if $b_i = a_i$, so a coordinate can only be small if the cube C has non-zero thickness in that coordinate. The point $t \in C$ is called *p-small* if it has at least p different coordinates that are small in this sense. Note that this is impossible if $p > \dim(C)$. The set of all p-small coordinates of C is denoted $\mathrm{Small}_p(C)$. We define $\mathrm{Large}_p(C)$ similarly, using the inequality $t_i > \frac{1}{2}(b_i + a_i)$ to define largeness of a coordinate t_i. An elementary observation is that, if $p + q > n$, then no point $t \in C$ can be simultaneously p-small and q-large, in other words:

$$\mathrm{Small}_p(C) \cap \mathrm{Large}_q(C) = \varnothing.$$

This observation will be key to the proof of the Blakers-Massey theorem.

3.12 Lemma: Let C be a cube, $f \colon C \to Z$ a continuous map, $Y \subseteq Z$ a subspace and p any integer in the range $1 \leqslant p \leqslant n = \dim(C)$. Assume that, for any proper face C_0 of C (*proper* means that $C_0 \neq C$) we have

$$f^{-1}(Y) \cap C_0 \ \subseteq\ \mathrm{Small}_p(C_0),$$

in other words, any point on C_0 that maps into Y under f is p-small in C_0. Then there is a homotopy $f \simeq g$ relative to ∂C such that $g^{-1}(Y) \subseteq \mathrm{Small}_p(C)$, in other words, any point of C that maps into Y under g is p-small in C. The same statement also holds with "small" replaced by "large".

- Proof.

3.13 Lemma: Let (X, A, B) be a (p, q)-connected triad for integers $p, q \geqslant 0$ and $f\colon [0,1]^n \to X$ a continuous map. By the Lebesgue lemma, we know that there exists a regular subdivision of the cube $[0,1]^n$ into subcubes, such that for each cube C of the subdivision either $f(C) \subseteq A$ or $f(C) \subseteq B$. By a *regular subdivision* we mean that we have a choice of $N \in \mathbb{N}$, and then we take the subdivision consisting of the cubes $\prod_{i=1}^{n} \left[\frac{k_i-1}{N}, \frac{k_i}{N}\right]$ for all $(k_1, \ldots, k_n) \in \{1, \ldots, N\}^n$. So we choose an $N \in \mathbb{N}$ such that the corresponding regular subdivision has the above property. Then there exists a homotopy $H\colon [0,1]^n \times [0,1] \longrightarrow X$ from f to g such that, for any cube C of the subdivision,

(I) · if $f(C) \subseteq A \cap B$, then $H|_{C\times\{t\}} = f|_C$ for all $t \in [0,1]$, i.e. H is constant on C,
· if $f(C) \subseteq A$, then $H(C \times [0,1]) \subseteq A$,
· if $f(C) \subseteq B$, then $H(C \times [0,1]) \subseteq B$,

(II) · if $f(C) \subseteq A$, then $g^{-1}(X \smallsetminus B) \cap C \subseteq \mathrm{Small}_{p+1}(C)$,
· if $f(C) \subseteq B$, then $g^{-1}(X \smallsetminus A) \cap C \subseteq \mathrm{Small}_{q+1}(C)$.

- Proof: next lecture.

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- Proof of Lemma 3.13.

3.14 Remark about where the hypothesis of (p, q)-connectivity is used in the proof of the Blakers-Massey theorem and in the proof of Lemma 3.13.

- Proof of Theorem 3.1 (*version I of the Blakers-Massey theorem*).

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3.15 Theorem: Let $A, B \subseteq X$ such that X is the union of the interiors of A and B, and similarly for $A', B' \subseteq X'$. Let $f\colon X \to X'$ be a continuous map such that $f(A) \subseteq A'$ and $f(B) \subseteq B'$. Suppose that $f|_A\colon A \to A'$ and $f|_B\colon B \to B'$ are n-connected and $f|_{A\cap B}\colon A \cap B \to A' \cap B'$ is $(n-1)$-connected. Then f is n-connected.

3.16 Corollary: a version of this theorem for arbitrary open covers, instead of covers by just two subsets.

- Proof.

3.17 Fact about closed cofibrations (NDR-pairs): Suppose that g is the pushout along a closed cofibration of a map f. If f is a homotopy equivalence, then so is g.
In particular, if the inclusion $A \hookrightarrow X$ is a closed cofibration and A is contractible, then the quotient map $X \to X/A$ is a homotopy equivalence.

3.18 Corollary: Let $f\colon X \to Y$ be a map between well-based spaces. If f is n-connected, then Σf is $(n+1)$-connected.

- Proof.
- Correction: in the statement of the Freudenthal suspension theorem (Theorem 3.8), X should be assumed to be well-based, so that in the proof we may replace the based suspension with the unbased suspension (without changing homotopy type).

3.19 Corollary: S^n is $(n-1)$-connected.

- (Of course, this also follows from the cellular approximation theorem.)

4. Representability theorems

4.1 Definition: the categories CW_*, $\mathrm{CW}_*^{\mathrm{fd}}$ and $\mathrm{CW}_*^{\mathrm{f}}$.

4.2 Theorem (Brown): Let $\mathcal{C}$ be one of the above three categories and let $T\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ be a homotopy-invariant functor satisfying the *wedge axiom* (W) and the *Mayer-Vietoris axiom* (MV). Also assume that T is not the functor sending every object to the empty set. In the case when $\mathcal{C} = \mathrm{CW}_*^{\mathrm{f}}$, assume that $T(S^n)$ is countable for all $n \geqslant 1$. Then there exists $Z \in \mathrm{CW}_*$ and a natural isomorphism

$$T \cong \langle - , Z\rangle\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}.$$

Moreover, if $\langle - , Z\rangle \cong \langle - , Z'\rangle$ for $Z, Z' \in \mathrm{CW}_*$, then $Z \simeq Z'$.

- NB: A homotopy-invariant functor T satisfying the wedge and Mayer-Vietoris axioms is called *half-exact*.

4.3 Remarks:

(1) When $\mathcal{C} = \mathrm{CW}_*$, the uniqueness statement follows from the Yoneda lemma.
(2) Remark about *half-exact* and *exact* functors.
(3) Reformulation of the (MV) axiom when T factors through the category of abelian groups.

4.4 Example: reduced singular homology in degree $k \geqslant 0$ with coefficients in A.
4.5 Definition: the *Eilenberg-MacLane space* $K(A, k)$.
4.6 Theorem (Adams): Take $\mathcal{C} = \mathrm{CW}_*^{\mathrm{f}}$ in Theorem 4.2, but instead of assuming that $T(S^n)$ is countable for all n, assume that T factors as

$$\mathrm{CW}_*^{\mathrm{f}} \xrightarrow{T'} \mathrm{Group} \longrightarrow \mathrm{Set}.$$

Then $T \cong \langle - , Z\rangle$ for some $Z \in \mathrm{CW}_*$. Moreover, $T' \cong \langle - , Z\rangle$ for some *weak H-group* Z.
4.7 Definition of *weak homotopy* and *weak H-group* in CW_*.
4.4 Example (cont.): the homotopy class $K(A, k)$ contains a weak H-group.
- We'll first prove Theorem 4.2 for $\mathcal{C} = \mathrm{CW}_*^{\mathrm{fd}}$.

4.8 Definition: *pro-universal sequence* (for a CW-complex Y and a functor T).
4.9 Observation: the functor T is represented by Y if and only if there exists a pro-universal sequence for (T, Y).

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- Fix a half-exact functor $T\colon (\mathrm{CW}_*^{\mathrm{fd}})^{\mathrm{op}} \longrightarrow \mathrm{Set}$.

4.10 Observations:
(a) For each $X \in \mathrm{CW}_*^{\mathrm{fd}}$, the set $T(X)$ is naturally based. In other words, T factors through the category of based sets.
(b) If X is a co-H-group, then $T(X)$ is a group. In particular, $T(S^n)$ is a group for all $n \geqslant 1$.

4.11 Lemma: Let $\theta\colon T \to U$ be a natural transformation between half-exact functors. Fix $n \geqslant 1$. Assume that $T(S^m) \to U(S^m)$ is a bijection for all $m \leqslant n-1$ and a surjection for $m = n$. Then, for all $X \in \mathrm{CW}_*^{\mathrm{fd}}$, the function $T(X) \to U(X)$ is a bijection if $\dim(X) \leqslant n-1$ and a surjection if $\dim(X) = n$.
4.12 Lemma: for any $Y \in \mathrm{CW}_*$, the functor $\langle - , Y\rangle\colon (\mathrm{CW}_*^{\mathrm{fd}})^{\mathrm{op}} \to \mathrm{Set}$ is half-exact.
- Proof of Theorem 4.2 (existence statement) for $\mathcal{C} = \mathrm{CW}_*^{\mathrm{fd}}$. (Construct, recursively on n, the n-skeleton Y^n of $Y \in \mathrm{CW}_*$ and a pro-universal sequence $u_n \in T(Y^n)$.)

4.13 Sublemma: the (MV) axiom applies not just to diagrams of inclusions of subcomplexes, but also more generally, for example to the pushout square that attaches n-cells to an $(n-1)$-dimensional CW-complex. (Proof uses Fact 3.17.)
4.14 Lemma: Let X, Y be pointed CW-complexes, let $u_n \in T(X^n)$ be a compatible sequence and let $v_n \in T(Y^n)$ be a pro-universal sequence. Then there exists a cellular map $f\colon X \to Y$ such that

$$T(f_n\colon X^n \to Y^n)(v_n) = u_n$$

for all n.
- Proof

4.15 Lemma: In the previous lemma, if u_n is also pro-universal, then f is a homotopy equivalence.
- Proof.

4.16 Theorem: Let T and U be half-exact functors $(\mathrm{CW}_*^{\mathrm{fd}})^{\mathrm{op}} \longrightarrow \mathrm{Set}$ and let X and Y be pointed CW-complexes. Suppose we have natural isomorphisms $\theta\colon \langle - , X\rangle \cong T$ and $\varphi\colon \langle - , Y\rangle \cong U$.
(a) If $\sigma\colon T \to U$ is any natural transformation, there is a cellular map $f\colon X \to Y$ such that the following square commutes:

$$\begin{tikzcd} T \arrow[r, "\sigma"] & U \\ \langle - , X\rangle \arrow[u, "\theta"] \arrow[r, "f \circ -"'] & \langle - , Y\rangle \arrow[u, "\varphi"'] \end{tikzcd} \tag{9}$$

(b) If σ is a natural isomorphism, then f is a homotopy equivalence.

- Note: this implies the uniqueness statement of Theorem 4.2 for $\mathcal{C} = \mathrm{CW}_*^{\mathrm{fd}}$.

14. Wednesday 21 November

- Proof of Theorem 4.16.

4.17 Remark about the proof in the case of $\mathcal{C} = \mathrm{CW}_*^{\mathrm{f}}$. The difficulty is that, in the construction of the CW-complex Y representing T, we need to attach n-cells indexed by the sets $T(S^n)$ and $\ker(\pi_{n-1}(Y^{n-1}) \to T(S^{n-1}))$. Even if we assume that $T(S^n)$ is finite for all n, the kernel may not be finite, even if Y^{n-1} is a finite complex. We can modify the construction to attach n-cells indexed just by a generating set of this kernel, but this still doesn't help since finite CW-complexes may have non-finitely-generated homotopy groups (the standard example is $\pi_2(S^1 \vee S^2)$).
The idea is instead to assume that the $T(S^n)$ are all countable, and, as a first (technical!) step, to show that T extends to the category of based, *countable* CW-complexes, satisfying the wedge axiom and a slightly weaker form of the Mayer-Vietoris axiom. Then we may proceed as before, attaching cells indexed by $T(S^n)$ and by $\ker(\pi_{n-1}(Y^{n-1}) \to T(S^{n-1}))$ to construct a CW-complex representing T. Here, we use the fact that the homotopy groups of countable CW-complexes are countable – a fact that follows by induction from the fact that the homotopy groups of spheres are countable.

4.18 Remark about the proof in the case of $\mathcal{C} = \mathrm{CW}_*$. The method for $\mathrm{CW}_*^{\mathrm{fd}}$ applies to give us a CW-complex Y and a natural isomorphism $\langle - , Y\rangle \cong T$ that is defined on the subcategory $\mathrm{CW}_*^{\mathrm{fd}} \subset \mathrm{CW}_*$. We therefore need to extend this.

4.19 Definition: a *universal element* $u \in T(Y)$ for a functor $T\colon (\mathrm{CW}_*)^{\mathrm{op}} \to \mathrm{Set}$.

4.20 Lemma: if $Y \in \mathrm{CW}_*$ and $T\colon (\mathrm{CW}_*)^{\mathrm{op}} \to \mathrm{Set}$ is half-exact, then

$$\lim_n(T(Y^n)) \twoheadleftarrow T(Y) \tag{10}$$

is surjective.

4.21 Corollary: there is a universal element $u \in T(Y)$.

- From this we get a natural transformation $\langle - , Y\rangle \to T$ that is defined on all CW-complexes, and which agrees with the previous natural isomorphism on finite-dimensional CW-complexes. However, we cannot directly use Lemma 4.20 to deduce that it is an isomorphism also for infinite-dimensional CW-complexes, since we only have surjectivity of (10), not injectivity. Counterexamples that show the failure of injectivity can be given by so-called *phantom maps* (examples coming soon).
- Instead, we use an infinite-dimensional analogue of Lemma 4.14.

4.22 Lemma: Let $T\colon (\mathrm{CW}_*)^{\mathrm{op}} \to \mathrm{Set}$ be half-exact, $v \in T(Y)$ a universal element and $u \in T(X)$ any element. Also let $A \subseteq X$ be a subcomplex and $f_0\colon A \to Y$ a based, cellular map such that $T(f_0)(v) = T(A \hookrightarrow X)(u)$. Then f_0 extends to a cellular map $f\colon X \to Y$ such that $T(f)(v) = u$.

- Proof of Theorem 4.2 for $\mathcal{C} = \mathrm{CW}_*$.

4.23 Examples of *phantom maps* and *weak phantom maps*. (See also optional exercise F.)

4.24 Definition: $\mathcal{C}$*-homotopy* and $\mathcal{C}$*-H-group* for any full subcategory $\mathcal{C} \subseteq \mathrm{CW}_*$.

4.25 Observation: if Y is a $\mathcal{C}$-H-group, then $\langle - , Y\rangle$ is a functor to the category of groups.

4.26 Theorem: Let $\mathcal{C}$ be one of CW_*, $\mathrm{CW}_*^{\mathrm{fd}}$ or $\mathrm{CW}_*^{\mathrm{f}}$, as in Theorem 4.2, and let

$$T\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Group}$$

be a half-exact functor. Then there is a $\mathcal{C}$-H-group Y and a natural isomorphism $\langle - , Y\rangle \cong T$.

- Sketch of proof of why Y has a structure of a $\mathcal{C}$-H-group.

15. Monday 26 November

4.27 Definition: *reduced cohomology theories* on the category cw_* of based CW-complexes (whose basepoint is a 0-cell) and based, continuous maps. (Two versions of the definition, either with a connecting homomorphism or with a suspension isomorphism.)

4.28 Remarks:

(a) These two definitions are equivalent.

(b) We could define this on the category Top_* instead of cw_*, and require that *weak* homotopy equivalences are taken to isomorphisms. By CW-approximation, such an object is determined by its restriction to $\mathrm{cw}_* \subset \mathrm{Top}_*$. By Whitehead's theorem, this restriction is exactly the type of object defined above. For example, singular cohomology extends in this way, but Čech cohomology does not.

4.29 Lemma: if $\{h^n\}$ is a cohomology theory, then each h^n is a half-exact functor (when restricted to $\mathrm{CW}_* \subset \mathrm{cw}_*$).

- Proof. (This is essentially part of the derivation of the Mayer-Vietoris sequence from the long exact sequence of a pair.)

4.30 Corollary: there is therefore a based, connected CW-complex Y_n such that $\langle -, Y_n\rangle \cong h^n|_{\mathrm{CW}_*}$.

4.31 Definition: *reduced homology theory* on the category cw_*.

4.32 Remarks on the axioms:

(a) As in the proof of Lemma 4.29 (in the case of cohomology), the exactness axiom implies the existence of Mayer-Vietoris sequences, which implies the wedge axiom for *finite* collections of objects.

(b) There is another axiom, called the *direct limit axiom* (DL). This axiom implies the wedge axiom (idea: use the direct limit axiom to break up a wedge into its finite subcomplexes, then use the *finite* wedge axiom inside the direct limit to rewrite the wedge as a direct sum). In fact, the direct limit axiom is equivalent to the wedge axiom (in the presence of the exactness axiom), but the other implication is more complicated to prove.

4.33 Definitions of (naive, or pre-)*spectrum*, *Ω-spectrum*, *infinite loopspace* as well as the homotopy groups of a spectrum.

- Side note: Theorem (Milnor): if Z is a zero-based CW-complex (based CW-complex whose basepoint is a 0-cell), then ΩZ is homotopy equivalent to a zero-based CW-complex.

4.34 Remark on the homotopy groups of an Ω-spectrum.

4.35 Example: the *suspension spectrum* of $X \in \mathrm{cw}_*$. Its homotopy groups are the *stable homotopy groups* of X.

- Note: one may define morphisms and 2-morphisms (homotopies) of spectra and Ω-spectra in the obvious ("naive") way, and therefore obtain 2-categories $\Omega\text{-Spec} \subseteq \text{Spec}$. We may also define morphsisms of (co)homology theories in the obvious way, as sequences of natural transformations commuting with the given suspensions isomorphisms, and so we have categories Coh and Hom that may be thought of as 2-categories by declaring all of their 2-morphisms to be identities.

4.36 Theorem: there is a 2-functor $\Omega\text{-Spec} \longrightarrow \text{Coh}$, given on objects by

$$\{Y_n, f_n\} \longmapsto (\langle - \,, Y_n\rangle, (f_n \circ -) \circ \Sigma),$$

that induces a *bijection* of equivalence classes of objects and *surjections* of equivalence classes of morphisms. In other words, every cohomology theory is represented by an Ω-spectrum that is unique up to homotopy equivalence, and every morphism between cohomology theories is represented by a morphism between the corresponding Ω-spectra (but this representing morphism is not unique up to homotopy in general).

4.37 Example: the *Eilenberg-MacLane Ω-spectrum HG*, for an abelian group G, represents singular cohomology with coefficients in G.

16. Wednesday 28 November

4.38 Definition: the *smash product* $X \wedge Y$ of $X \in \mathrm{cw}_*$ and a spectrum Y.

4.39 Theorem: there is a 2-functor $\text{Spec} \longrightarrow \text{Hom}$, given on objects by

$$Y \longmapsto \big(h_n(X) = \pi_n(X \wedge Y)\ ,\ \sigma_n(X) = \text{colimit of suspension homomorphisms}\big),$$

that induces a *bijection* of equivalence classes of objects and *surjections* of equivalence classes of morphisms. In other words, every homology theory is represented[7] by a spectrum that is

[7] This is an informal use of the word "represented".

unique up to homotopy equivalence, and every morphism between homology theories is represented by a morphism between the corresponding spectra (but this representing morphism is not unique up to homotopy in general).

- Note: For cohomology theories, we need to assume that Y is an Ω-spectrum in order to ensure that the suspension isomorphisms that we define really are isomorphisms. On the other hand, for homology theories, we do not need to assume this: the Freudenthal suspension theorem implies that the suspension isomorphisms, as defined, really are isomorphisms.

4.40 Example: the *Eilenberg-MacLane spectrum* HG, for an abelian group G, represents singular homology with coefficients in G via Theorem 4.39.

4.41 Construction: there is a 2-functor $\omega\colon \mathrm{Spec} \longrightarrow \Omega\text{-}\mathrm{Spec} \subseteq \mathrm{Spec}$ given on objects by

$$Y \longmapsto (\omega Y)_n = \text{the mapping telescope of the sequence } Y_n \to \Omega Y_{n+1} \to \Omega^2 Y_{n+2} \to \cdots$$

together with canonical weak equivalences $(\omega Y)_n \to \Omega(\omega Y)_{n+1}$ given by the up-to-homotopy universal property of mapping telescopes (homotopy colimits). (We should also either take a functorial CW-approximation of this as a second step, or appeal to the theorem of Milnor mentioned just after Definition 4.33 to ensure that this is a diagram of CW-complexes.)

- We may therefore associate a cohomology theory to any spectrum via

$$Y \longmapsto h^n(-) = \langle - \,, (\omega Y)_n \rangle.$$

4.42 Remarks:

(a) The functor ω is idempotent up to homotopy equivalence on objects, i.e., $\omega^2 Y \simeq \omega Y$.

(b) For any $X \in \mathrm{cw}_*$ and spectrum Y, we have $\pi_n(X \wedge Y) \cong \pi_n(X \wedge (\omega Y))$, so the homology theory associated to ωY is the same as that associated to Y. If we write $(-)_*$ for the 2-functor of Theorem 4.39, this says that $(\omega Y)_* \cong Y_*$ for all spectra Y.

(c) One can *directly* define a cohomology theory on the category $\mathrm{cw}_*^{\mathrm{f}}$ of *finite*, zero-based CW-complexes, without using ω. This is a 2-functor $\mathrm{Spec} \to \mathrm{Coh}^{\mathrm{f}}$ given on objects by

$$Y \longmapsto h^n(X) = \mathrm{colim}_i \langle \Sigma^i X \,, Y_{n+i} \rangle.$$

This formula satisfies the finite wedge axiom, but not the infinite wedge axiom, which is why we only obtain a cohomology theory on $\mathrm{cw}_*^{\mathrm{f}}$. This is compatible with Theorem 4.36 in the sense that the following square of 2-functors commutes:

$$\begin{tikzcd} \mathrm{Spec} \arrow[r] \arrow[d, "\omega"'] & \mathrm{Coh}^{\mathrm{f}} \\ \Omega\text{-}\mathrm{Spec} \arrow[r] & \mathrm{Coh} \arrow[u, "\text{restriction}"'] \end{tikzcd}$$

On equivalence classes of objects, the bottom horizontal map induces a bijection (by Theorem 4.36) and the map ω induces a surjection by remark (a) above.

4.43 Theorem (Adams): the 2-functor $\mathrm{Spec} \to \mathrm{Coh}^{\mathrm{f}}$ induces a bijection on equivalence classes of objects.

4.44 Corollary: all of the 2-functors in the square above induce bijections on equivalence classes of objects. For the right-hand vertical map, this means that every cohomology theory on $\mathrm{cw}_*^{\mathrm{f}}$ extends uniquely to a cohomology theory on cw_*.

4.45 Proposition: if $h \to k$ is a morphism of homology theories such that $h_n(S^0) \to k_n(S^0)$ is an isomorphism of groups for all n, then $h \to k$ is an isomorphism of homology theories, equivalently, $h_n(X) \to k_n(X)$ is an isomorphism of groups for all n and all $X \in \mathrm{cw}_*$.

- Sketch proof.

4.46 Remark: there is an exactly analogous result for cohomology theories.

4.47 Remarks:

(a) The sequence $\{h_n(S^0) \mid n \in \mathbb{Z}\}$ of abelian groups does *not* generally determine h up to isomorphism.

(b) However, it *does* if $h_n(S^0) = 0$ for all $n \neq 0$. (In this case, h must be isomorphic to ordinary singular homology with coefficients in the abelian group $h_0(S^0)$.) This follows from:

4.48 Lemma: if h and k are homology theories, then any group homomorphism $h_0(S^0) \to k_0(S^0)$ may be extended to a morphism $h \to k$ of homology theories.

- Remark: the existence of non-isomorphic homology theories with the same coefficient groups (i.e. values on the 0-sphere) means that the extension in Lemma 4.48 must be non-unique in general.

4.49 Remark about why homology theories cannot be "corepresentable", in the sense of a sequence of functors of the form $\langle Y_n\,, -\rangle$. This is because representable functors must take coproducts to products (which is true for cohomology theories by the wedge axiom), and corepresentable functors must preserve products, i.e. take products to products. This is *not* true for homology theories, as one can see already for ordinary singular homology: the homology of a product is not the product of the homologies (consider for example $H_2(S^1 \times S^1)$). The Künneth theorem states something analogous to preserving products, but with several differences, namely (a) considering all degrees of homology simultaneously as a functor into graded abelian groups, (b) using the smash product in cw_* rather than the categorical product (the direct product), and (c) most significantly, using the tensor product in the category of graded abelian groups, instead of the categorical product (which is the direct sum if the number of objects is finite).

- Recall Theorem 1.15(b) from lecture 1 (which implies the CW-approximation theorem). We will now use this to construct *Moore-Postnikov* towers of maps between spaces.
- Recall also Lemma 1.24, the Compression Lemma:

1.24 Lemma: Let (X, A) be a CW-pair and (Y, B) be any pair of spaces with $B \neq \varnothing$. Assume that $\pi_n(Y, B) = 0$ for all choices of basepoint of B whenever there is an n-cell in $X \smallsetminus A$. Then any map of pairs $(X, A) \to (Y, B)$ may be *compressed* into B, i.e., there is a homotopy rel. A to a map with image contained in B.

- Definition: Given a space X with a subspace $A \subseteq X$ that is a CW-complex, and an integer $n \geqslant 0$, we say that an *n-connected CW-model* for (X, A) is a CW-complex Y containing A as a subcomplex and a map $Y \to X$ relative to A (i.e., restricting to the identity on A), such that (a) all cells of $Y \smallsetminus A$ have dimension at least $n + 1$ (this implies that the inclusion $A \hookrightarrow Y$ is *n-connected*, namely that it induces isomorphisms on $\pi_{\leqslant n-1}$ and surjections on π_n) and (b) the map $Y \to X$ is *n-coconnected*, namely it induces isomorphisms on $\pi_{\geqslant n+1}$ and injections on π_n.

4.50 Corollary: Let $Y \to X$ be an n-connected CW-model for (X, A) and let $Y' \to X'$ be an n'-connected CW-model for (X', A'), with $n' \geqslant n$, and let $g\colon (X, A) \to (X', A')$ be any map of pairs. Then there is a map $(Y, A) \to (Y', A')$ such that the square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "g"] \\ Y' \arrow[r, "f'"] & X' \end{tikzcd}$$

commutes when restricted to A (equivalently, $h|_A = g|_A$) and commutes up to homotopy rel. A without restriction. This h is unique up to homotopy rel. A.

- Proof.

4.51 Corollary: any two n-connected CW-models for (X, A) are homotopy equivalent rel. A.

4.52 Definition: a *Moore-Postnikov tower* of a map $f\colon X \to Y$ between path-connected spaces is a diagram of the form

$$\begin{tikzcd} & \vdots \arrow[d] & \\ & Z_3 \arrow[d] \arrow[ddr] & \\ & Z_2 \arrow[d] \arrow[dr] & \\ X \arrow[uur] \arrow[ur] \arrow[r] \arrow[rr, bend right] & Z_1 \arrow[r] & Y \end{tikzcd}$$

where each map $X \to Z_n$ is n-connected, each map $Z_n \to Y$ is n-coconnected and each map $Z_n \to Z_{n-1}$ is a Hurewicz fibration whose fibre is $K(\pi_{n-1}(F_f), n-1)$, where F_f is the homotopy fibre of f.

4.53 Theorem: if $f\colon X \to Y$ is a map between path-connected spaces and X is a CW-complex, then there exists a unique (up to weak homotopy equivalence of diagrams) Moore-Postnikov tower for f. Moreover, if we wish, we can ensure that the maps $X \to Z_n$ are all cofibrations and the maps $Z_n \to Y$ are all fibrations.

- Proof (sketch).

4.54 A map f is *principal* if it is weakly homotopy equivalent (in the category of maps) to a map of the form $p(g)\colon F_g \to X$ for some map $g\colon X \to Y$, where F_g is the homotopy fibre of g and $p(g)\colon F_g \to X$ is the first step in the fibre sequence of g (see Definition 2.55). Equivalently, we may say that every map appearing in a fibre sequence, except the right-most one, is principal by definition, and then any map that is weakly homotopy equivalent to such a map is also principal. In other words, the defining property of a principal map is that its fibre sequence may be extended not just infinitely to the left, but also (at least) one step to the right.

- Note that being principal is a property of based maps between based spaces, but for path-connected spaces the choices of basepoints do not affect whether or not a map is principal.

4.55 Lemma: Let X be a path-connected space and $A \subseteq X$ a path-connected subspace, such that the inclusion $i\colon A \hookrightarrow X$ has homotopy fibre $F_i \simeq K(G,n)$ for some abelian group G and integer $n \geqslant 1$. Then i is principal if and only if the action of $\pi_1(A)$ on $\pi_{n+1}(X,A)$ is trivial.

- Brief outline of proof.
- Remark: Note that $\pi_{n+1}(X,A) \cong \pi_n(F_i) \cong G$. Also note that, if i is principal, then its fibre sequence may be extended by $X \to Z$ for some space Z, whose loopspace must be $K(G,n)$, so that Z must be $K(G,n+1)$, by the characterisation of Eilenberg-MacLane spaces in terms of their homotopy groups. The extended fibre sequence of i therefore has the form

$$\cdots \to K(G,n) \longrightarrow A \hookrightarrow X \longrightarrow K(G,n+1).$$

4.56 Corollary: in Theorem 4.53, if the action of $\pi_1(X)$ on $\pi_n(M_f,X)$ is trivial for all $n \geqslant 2$, then f admits a Moore-Postnikov tower consisting of *principal fibrations*.

- Remark: when this happens, this is much more useful than having just a Moore-Postnikov tower of fibrations, because it means that each map $Z_n \to Z_{n-1}$ in the tower is determined by a map $Z_{n-1} \to K(\pi_{n-1}(F_f),n)$ (by taking the first step of the fibre sequence of this map), in other words a cohomology class $k_{n-1} \in H^n(Z_{n-1};\pi_{n-1}(F_f))$. In principle, the sequence of cohomology classes $\{k_n \mid n \geqslant 1\}$ allows one to reconstruct the whole tower from Z_1.

4.57 Examples:

(a) Take $Y = *$. Then, for any connected CW-complex X, we have a unique tower

$$\cdots \to Z_n \longrightarrow Z_{n-1} \to \cdots \qquad \cdots \to Z_3 \longrightarrow Z_2$$

of spaces and fibrations with compatible maps from X. Let us re-index by $X_n = Z_{n+1}$. Then each $X \to X_n$ induces isomorphisms on $\pi_{\leqslant n}$ and $\pi_{\geqslant n+1}(X_n) = 0$. This is the *Postnikov tower* of X. We may assume that the fibrations are principal if and only if the action of $\pi_1(X)$ on $\pi_n(CX,X) \cong \pi_{n-1}(X)$ is trivial for all $n \geqslant 3$. This condition is satisfied if X is either *simple* or *aspherical* (all of its higher homotopy groups vanish). The tower may be extended by a principal fibration $X_1 \to X_0$ if and only if X is simple.

(b) Take $X = *$. Then, for any connected space Y, we have a unique tower

$$\cdots \to Z_n \longrightarrow Z_{n-1} \to \cdots \qquad \cdots \to Z_3 \longrightarrow Z_2 \longrightarrow Z_1$$

of spaces and principal fibrations equipped with compatible maps to Y. Each space Z_n is n-connected and the map $Z_n \to Y$ induces isomorphisms on $\pi_{\geqslant n+1}$. This is called the *Whitehead tower* of Y, and the map $Z_n \to Y$ is called the *n-connected cover* of Y. For example, we see by uniqueness that the map $Z_1 \to Y$ is (up to homotopy equivalence) the universal cover $\widetilde{Y} \to Y$. Another example is that, for $Y = S^2$, the 2-connected cover $Z_2 \to Y$ is the Hopf fibration $\eta\colon S^3 \to S^2$.

4.58 Lemma: Let X be a connected CW-complex and let $\{X_n\}$ be a Postnikov tower for X. There is then a well-defined map

$$X \longrightarrow \lim(\cdots \to X_n \to X_{n-1} \to \cdots \to X_2 \to X_1) \tag{11}$$

to the inverse limit of the tower. This map is a weak equivalence.

4.59 Definition: the cohomology classes $k_n \in H^{n+1}(Z_n; \pi_n(F_f))$, mentioned just after Corollary 4.56, in the special case when $Y = *$, corresponding to the Postnikov tower of X, are called the *k-invariants* of X. In this case f is the unique map from X to a point, so $F_f \simeq X$ and we can rewrite them as

$$k_n \in H^{n+1}(Z_n; \pi_n(X)).$$

The stage X_1 in the Postnikov tower for X is $K(\pi_1(X), 1)$. Thus, by the discussion just after Corollary 4.56, together with Lemma 4.58, every connected, semisimple[8] CW-complex X is determined up to weak homotopy equivalence by $\pi_1(X)$ together with the cohomology classes $\{k_n \mid n \geqslant 1\}$. Of course, the class k_n only becomes *defined* once the previous classes $k_1, \ldots, k_{n-1}$ have been determined, so that the space Z_n is known.

4.60 Application: Let (Y, A) be a CW-pair and let X be a connected, simple CW-complex. Assume that $H^{n+1}(Y, A; \pi_n(X)) = 0$ for all $n \geqslant 1$. Then every map $A \to X$ extends to $Y \to X$.

- Sketch of proof: The idea is to start with the trivial map $Y \to X_0 = *$ and try to lift it to X_n for all n. Then there is just a small extra argument to lift it along the map (11). The problem of lifting a map from X_{n-1} to X_n may be rewritten, using the k-invariants, as the vanishing of a certain cohomology class. Since we are doing everything relative to a given map $A \to X \to X_n$, this turns out to be a *relative* cohomology class, living in the group that we have assumed to be zero.
- NB: the reason that we need X to be simple and not just semisimple is that we need its Postnikov tower of principal fibrations to start from X_0, and not only from X_1.

4.61 Corollary (also 1.31): If $f\colon X \to Y$ is a map between connected, simple CW-complexes that induces isomorphisms on integral homology in every degree, then f is a homotopy equivalence.

- Proof. We may assume that $X \hookrightarrow Y$ is an inclusion, by replacing Y with M_f if necessary. We know that $H_*(Y, X) = 0$ and by Whitehead's theorem it will suffice to show that $\pi_*(Y, X) = 0$. This will follow from the relative Hurewicz theorem as long as the action of $\pi_1(X)$ on $\pi_n(Y, X)$ is trivial for all n. We know by hypothesis that the action of $\pi_1(X)$ on $\pi_n(X)$ and on $\pi_n(Y)$ is trivial for all n, but this on its own (using the long exact sequence of relative homotopy groups of (Y, X)) is *not* enough to deduce that the action on $\pi_n(Y, X)$ is also trivial. However, it will be enough if we know that $\pi_n(Y) \to \pi_n(Y, X)$ is surjective for all n. By exactness, this is equivalent to the claim that $\pi_n(X) \to \pi_n(Y)$ is injective for all n. For this to be true, it suffices for the inclusion $X \hookrightarrow Y$ to admit a left-inverse. In other words, we need to extend the identity $X \to X$ to Y. The fact that such an extension exists now follows from Application 4.60 above: the condition that certain relative cohomology groups vanish is implied via the universal coefficient theorem by the fact that $H_*(Y, X) = 0$.

17. Monday 3 December

5. Quasifibrations and the Dold-Thom theorem

- Lecture given by Benjamin Böhme.
- Reference: A. Dold, R. Thom, *Quasifaserungen und Unendliche Symmetrische Produkte.* Annals of Mathematics (2) 67.2, pp. 239–281, (1958).
- Link for the notes: mdp.ac/teaching/18-algebraic-topology/outline-of-lectures-17-and-18.pdf

18. Monday 10 December

- Lecture given by Benjamin Böhme.
- Reference: A. Dold, R. Thom, *Quasifaserungen und Unendliche Symmetrische Produkte.* Annals of Mathematics (2) 67.2, pp. 239–281, (1958).
- Link for the notes: mdp.ac/teaching/18-algebraic-topology/outline-of-lectures-17-and-18.pdf

19. Monday 17 December

6. Serre classes and rational homotopy groups of spheres

[8] This is an *ad hoc* terminology meaning that the action of $\pi_1(X)$ on $\pi_n(X)$ is trivial for all $n \geqslant 2$. Simple spaces are semisimple, and aspherical spaces are also semisimple.

6.1 Definition: Fix a commutative ring R and a class $\mathcal{C}$ of R-modules.

- $\mathcal{C}$ is a *Serre class* if, whenever
$$0 \to A \to B \to C \to 0$$
is an exact sequence of R-modules, we have: $\quad B \in \mathcal{C} \Leftrightarrow A \in \mathcal{C}$ and $C \in \mathcal{C}$.
- $\mathcal{C}$ is *saturated* if $A \in \mathcal{C}$ implies that $\bigoplus_{i \in I} A \in \mathcal{C}$ for any set I.
(Note that for finite I this is automatic if $\mathcal{C}$ is a Serre class.)
- A homomorphism $f\colon A \to B$ is a $\mathcal{C}$*-monomorphism* if and only if $\ker(f) \in \mathcal{C}$ and a $\mathcal{C}$*-epimorphism* if and only if $\operatorname{coker}(f) \in \mathcal{C}$. It is a $\mathcal{C}$*-isomorphism* if and only both $\ker(f)$ and $\operatorname{coker}(f)$ are in $\mathcal{C}$.

6.2 Examples of Serre classes:

(a) $\{0\}$ (saturated)
(b) {finite R-modules} (not saturated)
(c) {noetherian R-modules} (not saturated)
An R-module M is *noetherian* if every submodule of M is finitely generated. If the ring R itself is noetherian (as a module over itself), then an R-module M is noetherian if and only if it is finitely generated. Note that every principal ideal domain R is a noetherian ring, so in this case the Serre class under consideration is {finitely generated R-modules}. In particular, for $R = \mathbb{Z}$ we have the Serre class {finitely generated abelian groups}.
(d) {R-modules that are finitely generated as $\mathbb{Z}$-modules} (not saturated)
(e) {all R-modules} (saturated)
(f) If R is an integral domain, then we have the Serre class:
{torsion R-modules} (saturated)
An R-module M is *torsion* if, for every $m \in M$, there is some $\lambda \in R$ with $\lambda \neq 0$ and $\lambda m = 0$. In particular we have the Serre class {torsion abelian groups}.
(g) In general, intersections of Serre classes are again Serre classes. So, for example, if R is a noetherian integral domain (e.g. a principal ideal domain), we have the Serre class {finitely generated torsion R-modules} (not saturated)

In particular, we will write:

- $\mathcal{F} = \{$finite abelian groups$\}$
- $\mathcal{G} = \{$finitely generated abelian groups$\}$
- $\mathcal{T} = \{$torsion abelian groups$\}$

Note that $\mathcal{T} \cap \mathcal{G} = \mathcal{F}$. The Serre class $\mathcal{T}$ is saturated, but $\mathcal{G}$ and $\mathcal{F}$ are not.
For the proof of the finiteness (with a few exceptions) of the homotopy groups of spheres, the important Serre classes to consider will be $\mathcal{G}$ and $\mathcal{T}$.

6.3 Proposition: Homological algebra works "as expected" modulo $\mathcal{C}$, in particular we have:

(a) The two 4-lemmas (and hence the 5-lemma) hold with "epi" and "mono" replaced with "$\mathcal{C}$-epi" and "$\mathcal{C}$-mono" respectively.
(b) If f and g are both $\mathcal{C}$-mono (resp. $\mathcal{C}$-epi), then so is their composition.
(c) If two of f, g and $f \circ g$ are $\mathcal{C}$-isomorphisms, then so is the third.
(d) Let f be a homomorphism that is the colimit of a "ladder diagram" of homomorphisms f_n. If every f_n is a $\mathcal{C}$-mono (resp. $\mathcal{C}$-epi), then so is f.

6.4 Notation: For this lecture, fix a commutative ring R and an R-module M, and write H_* for the homology $H_*(-; M)$ with coefficients in M.

6.5 Lemma: Consider the attaching diagram

$$\begin{tikzcd} S^{t-1} \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook] \arrow[dl, phantom, "\ulcorner", very near start] \\ D^t \arrow[r, "\Phi"'] & B, \end{tikzcd}$$

in other words $B = A \cup_\varphi D^t$ is obtained from A by attaching a t-cell along the map φ, and Φ is the resulting characteristic map of this t-cell. Let $X \to B$ be a fibration. This induces three other fibrations
$$X' \to A \qquad Y \to D^t \qquad Y' \to S^{t-1}$$

by pulling back along the maps in this square. Then the induced map

$$H_*(Y,Y') \longrightarrow H_*(X,X')$$

is an isomorphism.

- Proof.

6.6 Remark: By the same argument, if we simultaneously attach many t-cells:

$$\begin{tikzcd} \coprod_i S_i^{t-1} \arrow[r, "\varphi"] \arrow[d, hook] & A \arrow[d, hook] \\ \coprod D_i^t \arrow[r, "\Phi"'] & B, \end{tikzcd} \tag{12}$$

then a fibration $X \to B$ pulls back to fibrations $X' \to A$ and

$$Y_i \to D_i^t \qquad Y_i' \to S_i^{t-1}$$

for each i. Then the induced map

$$\bigoplus_i H_*(Y_i,Y_i') \longrightarrow H_*(X,X')$$

is an isomorphism.

6.7 Fact: If $h\colon Z \to Y$ is a fibration and $f \simeq g\colon X \to Y$ are homotopic maps, then the two pulled-pack fibrations $f^*(Z) \to X$ and $g^*(Z) \to X$ are homotopy equivalent in Top/X.
Corollary: If $h\colon Z \to Y$ is a fibration and Y is contractible, then h is homotopy equivalent in Top/Y to the trivial fibration $Y \times h^{-1}(*) \to Y$.
(Proof: apply the fact to $f = \mathrm{id}_Y$ and $g = \mathrm{const}_Y$, the constant map to the basepoint of Y.)

6.8 Notation: For a path-connected space Z, write $Z \in \mathcal{C}(r,M)$ to mean that $H_i(Z;M) \in \mathcal{C}$ for all $1 \leqslant i < r$.

6.9 Proposition: In the setting of Remark 6.6, assume that B and $F = p^{-1}(*)$ are path-connected. If $F \in \mathcal{C}(r,M)$, then the map $H_*(X,X') \longrightarrow H_*(B,A)$ is:
   (a) surjective,
   (b) injective if $* \leqslant t$,
   (c) a $\mathcal{C}$-monomorphism if $* > t$ and $* - t < r$, and either
      (i) $\mathcal{C}$ is saturated, or
      (ii) the disjoint union $\coprod_i$ in (12) is finite.

- Proof, using Remark 6.6 and Fact 6.7.

6.10 Definition of *relative CW-complex* (same as that of *CW-complex*, except that we start by attaching 0-cells to an arbitrary space, instead of the empty set).

6.11 Proposition (*Relative CW approximation*): Let (X,A) be a k-connected pair for $k \geqslant -1$.
   (a) There exists a relative CW-complex (B,A) with only cells of dimension $\geqslant k+1$ and a weak equivalence $B \to X$ that restricts to the identity on A.
   (b) If X and A are path-connected, $\pi_1(X)$ is finitely generated and $H_*(X,A;\mathbb{Z})$ is finitely generated in degrees $* \leqslant \ell$, then we may assume that the relative CW-complex (B,A) has only *finitely many* cells of dimension $\leqslant \ell$.

- Sketch of proof, using the relative Hurewicz theorem (1.30) for part (b).

6.12 Theorem (*Fibration Theorem I*): Let (B,A) be a relative CW-complex with cells only in dimensions $\geqslant s$ (where $s \geqslant 1$) and let $p\colon X \to B$ be a fibration. Assume also that A, B and $F = p^{-1}(*)$ are path-connected, and that $F \in \mathcal{C}(r,M)$.
   (a) If $\mathcal{C}$ is saturated, then
   $$H_*(X,p^{-1}(A)) \longrightarrow H_*(B,A) \tag{13}$$
   is a $\mathcal{C}$-isomorphism in degrees $* \leqslant r+s-1$ and a $\mathcal{C}$-epimorphism in degrees $* \leqslant r+s$.
   (b) If (B,A) has only finitely many cells in dimensions $\leqslant d$, then (13) is a $\mathcal{C}$-isomorphism in degrees $* \leqslant \min(r+s-1,d+1)$ and a $\mathcal{C}$-epimorphism in degrees $* \leqslant \min(r+s,d+2)$.

- Proof, using Proposition 6.9 and the property (6.3)(d) of passing to colimits modulo $\mathcal{C}$.

6.13 Corollary (*Fibration Theorem II*): Let (B, A) be an $(s-1)$-connected pair and $p\colon X \to B$ be a fibration. Assume that A, B and $F = p^{-1}(*)$ are path-connected and that $F \in \mathcal{C}(r, M)$.

(a) If $\mathcal{C}$ is saturated, then
$$H_*(X, p^{-1}(A)) \longrightarrow H_*(B, A) \tag{14}$$
is a $\mathcal{C}$-isomorphism in degrees $* \leqslant r+s-1$ and a $\mathcal{C}$-epimorphism in degrees $* \leqslant r+s$.

(b) If $\pi_1(B)$ is finitely generated and $H_*(B, A; \mathbb{Z})$ is finitely generated in degrees $* \leqslant d$, then (14) is a $\mathcal{C}$-isomorphism in degrees $* \leqslant \min(r+s-1, d+1)$ and a $\mathcal{C}$-epimorphism in degrees $* \leqslant \min(r+s, d+2)$.

- Proof: this follows from Theorem 6.12 after taking a relative CW approximation as given in Proposition 6.11.

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6.14 Corollary: Let $\mathcal{C}$ be a Serre class of R-modules and recall that we denote the Serre class of finitely generated abelian groups by $\mathcal{G}$. Let $p\colon E \to B$ be a fibration with B and $F = p^{-1}(*)$ path-connected.
We then have the following implications, which say (approximately) that if two of the three spaces F, E, B have the property that their homology is contained in $\mathcal{C}$ in a range of degrees, then so does the third.

(E) If either $\mathcal{C}$ is saturated, or B is simply-connected and $B \in \mathcal{G}(r-1, \mathbb{Z})$, then:
$$F \in \mathcal{C}(r, M) \text{ and } B \in \mathcal{C}(r, M) \qquad \Rightarrow \qquad E \in \mathcal{C}(r, M).$$

(B) If either $\mathcal{C}$ is saturated, or B is simply-connected and $B \in \mathcal{G}(r, \mathbb{Z})$, then:
$$F \in \mathcal{C}(r, M) \text{ and } E \in \mathcal{C}(r+1, M) \qquad \Rightarrow \qquad B \in \mathcal{C}(r+1, M).$$

(F) If B is simply-connected and either $\mathcal{C}$ is saturated or $B \in \mathcal{G}(r, \mathbb{Z})$, then:
$$B \in \mathcal{C}(r+1, M) \text{ and } E \in \mathcal{C}(r, M) \qquad \Rightarrow \qquad F \in \mathcal{C}(r, M).$$

- Proof: follows from Corollary 6.13 and the long exact homology sequence for the pair (E, F).

6.15 Corollary: Let B be simply-connected.

(a) If $\mathcal{C}$ is saturated, then: $\Omega B \in \mathcal{C}(r, M) \Longleftrightarrow B \in \mathcal{C}(r+1, M)$.

(b) For $\mathcal{C} = \mathcal{G}$ we also have: $\Omega B \in \mathcal{G}(r, \mathbb{Z}) \Longleftrightarrow B \in \mathcal{G}(r+1, \mathbb{Z})$.

- Proof: using Corollary 6.14 applied to the path fibration $PB \to B$.

6.16 Proposition: Let $A \in \mathcal{G}$. Then $K(A, n) \in \mathcal{G}(\infty, \mathbb{Z})$ for all n. In other words: for all n and all $i \geqslant 1$, the homology groups $H_i(K(A, n); \mathbb{Z})$ are finitely generated.

- Proof: By Corollary 6.15 and the fact that $\Omega K(A, n) \simeq K(A, n-1)$, it is enough to prove that $K(A, 1) \in \mathcal{G}(\infty, \mathbb{Z})$, i.e., that the integral homology of $K(A, 1)$ is finitely generated in each degree. Using cellular homology, it therefore suffices to find a model for $K(A, 1)$ that admits a CW structure with only finitely many cells in each dimension. By the classification of finitely generated abelian groups and the fact that $K(-, 1)$ commutes with products, it suffices to show that $K(\mathbb{Z}, 1)$ and of $K(\mathbb{Z}/d, 1)$ have models admitting such a CW structure, for each $d \geqslant 2$. For $K(\mathbb{Z}, 1)$, we may of course take S^1. For $K(\mathbb{Z}/d, 1)$, we may take the orbit space $S^\infty/(\mathbb{Z}/d)$, where S^∞ is the unit sphere in $\mathbb{C}^\infty$ and $\mathbb{Z}/d$ acts by rotation by $2\pi/d$ in each coordinate. This space admits a CW structure with a single cell in each dimension.

- Now we will work towards calculating the rational cohomology of $K(\mathbb{Z}, n)$ for all n, which we will need later to study the finiteness of the homotopy groups of spheres.

6.17 Corollary: Let $f\colon X \to Y$ be a map between path-connected spaces. If $\widetilde{H}_*(F_f; \mathbb{Q}) = 0$, then f induces an isomorphism $H_*(X; \mathbb{Q}) \to H_*(Y; \mathbb{Q})$.

- Proof: using the Fibration Theorem (6.13) applied to the fibration $(N_f, F_f) \to (Y, *)$.

6.18 Proposition (*Wang sequence*): Let $p\colon X \to S^n$ be a fibration, $n \geqslant 2$, and write $F = p^{-1}(*)$. Then there is a long exact sequence (for any cohomology theory h):
$$\cdots \longrightarrow h^*(X) \xrightarrow{\text{restr.}} h^*(F) \xrightarrow{\theta} h^{*-n+1}(F) \longrightarrow h^{*+1}(X) \longrightarrow \cdots .$$

- Construction: First, use Fact 6.7 to trivialise the restrictions of p to the upper and lower hemispheres D^n_+ and D^n_- of S^n, and then carefully analyse the Mayer-Vietoris sequence of the covering $\{p^{-1}(D^n_+), p^{-1}(D^n_-)\}$ of X.

6.19 Remark: The proof of 6.18 shows that the map $\theta\colon h^*(F) \to h^{*-n+1}(F)$ has the following *derivation* property: for any $x, y \in h^*(F)$,

$$\theta(xy) = \theta(x)y + (-1)^{|x|(n-1)} x\theta(y). \tag{15}$$

6.20 Definition: Fix a ring R and an integer $n \geqslant 1$. Let $p\colon (E, E') \to B$ be a relative fibration over a path-connected space B (i.e. $p\colon E \to B$ is a fibration and its restriction to $E' \to B$ is also a fibration). Write $F = p^{-1}(*)$ and $F' = F \cap E'$ and abbreviate $H^*(-;R)$ to $H^*(-)$. Suppose that $H^n(F, F') \cong R$ and $H^i(F, F') = 0$ for $i \neq n$. Then a *Thom class* for p is an element $\tau \in H^n(E, E')$ such that, for each point $b \in B$, the pullback of τ is a generator for $H^n(p^{-1}(b), p^{-1}(b) \cap E')$ as an R-module.

6.21 Theorem (*Thom isomorphism theorem*):

(a) A Thom class for p exists if and only if the action of $\pi_1(B)$ on $H^n(F, F')$ is trivial. The action of an element $[\gamma] \in \pi_1(B)$ is defined as follows. Choose a representative loop $\gamma\colon [0,1] \to B$ and write $(\gamma^*(E), \gamma^*(E'))$ for the relative fibration over $[0,1]$ given by pulling back (E, E') along γ. Restricting to fibres over 0 and over 1 induces isomorphisms

$$H^n(F, F') \longleftarrow H^n(\gamma^*(E), \gamma^*(E')) \longrightarrow H^n(F, F'),$$

and $[\gamma]$ acts on $H^n(F, F')$ via this zig-zag of isomorphisms.

(b) If a Thom class τ exists, then the homomorphism

$$H^i(B) \longrightarrow H^{i+n}(E, E')$$

given by $x \mapsto p^*(x) \cup \tau$ is an isomorphism.

6.22 Proposition (*Gysin sequence*): As above, let $p\colon (E, E') \to B$ be a relative fibration over a path-connected space B. Let $n \geqslant 1$ and abbreviate $H^*(-;R)$ to $H^*(-)$. Assume that:

(i) $H^n(F, F') \cong R$ and $H^i(F, F') = 0$ for $i \neq n$,

(ii) the action of $\pi_1(B)$ on $H^n(F, F')$ is trivial,

(iii) $p^*\colon H^*(B) \to H^*(E)$ is an isomorphism in all degrees.

Then there is an exact sequence:

$$\cdots \longrightarrow H^{i-1}(E') \longrightarrow H^{i-n}(B) \xrightarrow{-\cup e} H^i(B) \xrightarrow{(p|_{E'})^*} H^i(E') \longrightarrow \cdots,$$

for a certain class $e \in H^n(B)$ (which is then called the *Euler class*).

- Proof: this follows easily from the Thom isomorphism theorem (6.21).

6.23 Example: Let $E \to B$ be a fibre bundle over a path-connected space with fibres homeomorphic to S^{n-1}. We may then take the *fibrewise cone* of E to obtain a relative fibre bundle

$$\big(\mathrm{Cone}^{\mathrm{fib}}(E), E \times [0,1)\big) \longrightarrow B,$$

whose relative fibres are homeomorphic to (D^n, S^{n-1}). Then, as long as the action of $\pi_1(B)$ on $H^n(D^n, S^{n-1})$ is trivial (for example if $\pi_1(B) = 0$ or $R = \mathbb{Z}/2$), we have a Gysin sequence of the form:

$$\cdots \longrightarrow H^{i-1}(E) \longrightarrow H^{i-n}(B) \longrightarrow H^i(B) \longrightarrow H^i(E) \longrightarrow \cdots.$$

6.24 Theorem (*Rational cohomology of integral Eilenberg-MacLane spaces*): Let $n \geqslant 1$.

(a) $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong H^*(S^n; \mathbb{Q})$ when n is odd.

(b) $H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}[\iota_n]$ when n is even,
(polynomial algebra generated by one element ι_n in degree n)

- Proof: Choose any representative $f\colon S^n \to K(\mathbb{Z}, n)$ of a generator of $\pi_n(K(\mathbb{Z}, n))$. By the Hurewicz theorem (and universal coefficient theorem), the induced map on $H^n(-;\mathbb{Q})$ is an isomorphism. Let ι_n be any non-zero element of $H^n(K(\mathbb{Z}, n); \mathbb{Q}) \cong \mathbb{Q}$.
- The proof is by induction on n. The case $n = 1$ is clear since $K(\mathbb{Z}, 1) \simeq S^1$, so we assume that $n \geqslant 2$.

- First assume that n is even. Let $E = \mathrm{Map}([0,1], K(\mathbb{Z},n))$ and $E' = \{\gamma \in E \mid \gamma(0) = *\} \subseteq E$. Note that E' is contractible. Then

$$(E, E') \longrightarrow K(\mathbb{Z}, n), \tag{16}$$

defined by $\gamma \mapsto \gamma(1)$, is a relative fibration whose relative fibre over the basepoint $* \in K(\mathbb{Z},n)$ is $(PK(\mathbb{Z},n), \Omega K(\mathbb{Z},n))$. Since $PK(\mathbb{Z},n)$ is contractible and $\Omega K(\mathbb{Z},n) \simeq K(\mathbb{Z},n-1)$, the inductive hypothesis implies that this relative fibration satisfies condition (i) of Proposition 6.22 (with $R = \mathbb{Q}$). Conditions (ii) and (iii) also hold since $\pi_1(K(\mathbb{Z},n)) = 0$ and the fibre $PK(\mathbb{Z},n)$ is contractible. Hence we have a Gysin sequence in rational cohomology:

$$\cdots \longrightarrow H^{i+n-1}(*) \longrightarrow H^i(K(\mathbb{Z},n)) \xrightarrow{-\cup e} H^{i+n}(K(\mathbb{Z},n)) \longrightarrow H^{i+1}(*) \longrightarrow \cdots,$$

where $e \in H^n(K(\mathbb{Z},n))$ is the Euler class of the relative fibration (16). This implies that the map $-\cup e$ is an isomorphism for $i \geqslant n$. In particular, $e \neq 0$ (since otherwise this would fail for $i = n$), so we may assume that $e = \iota_n$ (the Euler class is only well-defined up to multiplication by an element of $R^\times = \mathbb{Q} \smallsetminus \{0\}$, and $H^n(K(\mathbb{Z},n)) \cong \mathbb{Q}$). So we know that:
  - In degrees $i \leqslant n$, the only non-zero rational cohomology of $K(\mathbb{Z},n)$ is in degree zero and in degree n, where $H^n(K(\mathbb{Z},n)) \cong \mathbb{Q}\{\iota_n\}$.
  - In degrees $i \geqslant n$, the map $-\cup \iota_n \colon H^i(K(\mathbb{Z},n)) \to H^{i+n}(K(\mathbb{Z},n))$ is an isomorphism.

  These two facts imply by induction on the degree that $H^*(K(\mathbb{Z},n)) \cong \mathbb{Q}[\iota_n]$.
- Now assume that n is odd. Our aim is to show that f induces an isomorphism on rational cohomology in all degrees (we already know that it does in degrees $\leqslant n$). By Corollary 6.17 and the universal coefficient theorem, it will be enough to show that the reduced rational homology of the homotopy fibre F_f vanishes. Consider the path fibration

$$PK(\mathbb{Z},n) \longrightarrow K(\mathbb{Z},n)$$

and its pullback $F_f \to S^n$ along the map f. These both have fibre $\Omega K(\mathbb{Z},n) \simeq K(\mathbb{Z},n-1)$. An argument with the long exact sequences of these fibrations implies that F_f is n-connected. The Hurewicz theorem and the universal coefficient theorem then imply that the reduced rational cohomology of F_f vanishes in degrees $\leqslant n$ too. The Wang sequence of the fibration $F_f \to S^n$ is

$$\cdots \longrightarrow H^i(F_f) \longrightarrow H^i(K(\mathbb{Z},n-1)) \xrightarrow{\theta} H^{i-n+1}(K(\mathbb{Z},n-1)) \longrightarrow H^{i+1}(F_f) \longrightarrow \cdots,$$

so we know that θ is an isomorphism for $i = n-1$. If we can show that θ is an isomorphism for all $i \geqslant n-1$, then we will be done, because this implies that the reduced rational cohomology of F_f vanishes (and therefore its reduced rational homology vanishes too). By the inductive hypothesis, we know that $H^*(K(\mathbb{Z},n-1)) \cong \mathbb{Q}[\iota_{n-1}]$, so it will be enough to show, for all $k \geqslant 1$, that

$$\theta(\iota_{n-1}^k) = \lambda_k \, \iota_{n-1}^{k-1},$$

for some non-zero scalars $\lambda_k \in \mathbb{Q} \smallsetminus \{0\}$ depending on k. We already know that this is true for $k = 1$.
- Claim: for all $k \geqslant 1$, we have $\theta(\iota_{n-1}^k) = k\lambda_1 \iota_{n-1}^{k-1}$.
- Proof by induction on k. We already know the base case $k = 1$, so assume that $k \geqslant 2$ and write $\iota_{n-1} = e$ for short. Then, using (15), we have:

$$\theta(e^k) = \theta(e \cup e^{k-1}) = \lambda_1 e^{k-1} + e \cup (k-1)\lambda_1 e^{k-2} = k\lambda_1 e^{k-1}.$$

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- Recall from Definition 6.1 the notions of *Serre class*, and of *saturated Serre class*. Another property of Serre classes that we will be interested in is the following.
- Definition: A Serre class of R-modules $\mathcal{C}$ is called *homology-closed* if $A \in \mathcal{C}$ implies that $K(A,1) \in \mathcal{C}(\infty, R)$, in other words, $H_i(K(A,1); R) \in \mathcal{C}$ for all $i \geqslant 1$.
- Examples: the examples of Serre classes of $\mathbb{Z}$-modules that we will be interested in are:

- $\mathcal{G} = \{$finitely generated abelian groups$\}$
- $\mathcal{T} = \{$torsion abelian groups$\} = \{$abelian groups A such that $A \otimes \mathbb{Q} = 0\}$
- $\mathcal{F} = \mathcal{G} \cap \mathcal{T} = \{$finite abelian groups$\}$

- The class $\mathcal{T}$ is saturated, whereas $\mathcal{G}$ and $\mathcal{F}$ are not.
- The classes $\mathcal{G}$ and $\mathcal{F}$ are homology-closed (for $\mathcal{G}$ this is part of Proposition 6.10, and for $\mathcal{F}$ it can be shown similarly). Also:

6.25 Lemma: $\mathcal{T}$ is homology-closed.

- Proof: A general fact from group homology is that, for any abelian group A,

$$H_*(A;\mathbb{Q}) \;\cong\; \textstyle\bigwedge^*(A \otimes \mathbb{Q}),$$

the exterior algebra over $\mathbb{Q}$ generated by the vector space $A \otimes \mathbb{Q}$ in degree 1. (See for example Theorem V.6.4 of [Brown, *Cohomology of groups*].) Suppose that $A \in \mathcal{T}$. Then, by the above fact and the universal coefficient theorem,

$$H_*(A;\mathbb{Z}) \otimes \mathbb{Q} \cong H_*(A;\mathbb{Q}) \cong \textstyle\bigwedge^*(A \otimes \mathbb{Q}) = \bigwedge^*(0) = \mathbb{Q}[0],$$

where $\mathbb{Q}[0]$ denotes the graded $\mathbb{Q}$-algebra consisting just of $\mathbb{Q}$ in degree zero. Thus, for every $i \geqslant 1$, $H_i(A;\mathbb{Z}) \cong H_i(K(A,1);\mathbb{Z})$ is torsion.

6.26 Theorem (*The mod-$\mathcal{C}$ Hurewicz theorem*):
Let $\mathcal{C}$ be a homology-closed Serre class of $\mathbb{Z}$-modules and X a 1-connected space. Let $n \geqslant 2$ and assume that:
(i) either $\mathcal{C}$ is saturated
(ii) or $X \in \mathcal{G}(n,\mathbb{Z})$, i.e., $H_i(X;\mathbb{Z})$ is finitely generated for all $i < n$.
Then:

$$\Pi(n) \Longleftrightarrow \mathrm{H}(n) \Longrightarrow \mathrm{I}(n),$$

where:

$$\begin{aligned} \Pi(n) &= \pi_i(X) \in \mathcal{C} \text{ for all } i < n \\ \mathrm{H}(n) &= H_i(X;\mathbb{Z}) \in \mathcal{C} \text{ for all } i < n \\ \mathrm{I}(n) &= \text{The Hurewicz map } \pi_i(X) \to H_i(X;\mathbb{Z}) \text{ is a } \mathcal{C}\text{-isomorphism for all } i \leqslant n \end{aligned}$$

- Proof: next lecture.

6.27 Corollary: If X is a 1-connected CW-complex with only finitely many cells in each dimension, then $\pi_i(X)$ is finitely generated for all i.

- Proof: The assumption implies that $H_i(X;\mathbb{Z})$ is finitely generated for all i, by considering cellular homology. Then Theorem 6.26 with $\mathcal{C} = \mathcal{G}$ implies the result.

6.28 Corollary (*The mod-$\mathcal{C}$ Whitehead theorem*):
Let $\mathcal{C}$ be a saturated, homology-closed Serre class of $\mathbb{Z}$-modules and let $f\colon X \to Y$ be a map between simply-connected spaces such that F_f is also simply-connected (equivalently, $\pi_2(f)$ is a surjection). Then the following two statements are equivalent.
(1) $\pi_i(f)$ is a $\mathcal{C}$-isomorphism for all $i < n$ and a $\mathcal{C}$-epimorphism for $i = n$.
(2) $H(f;\mathbb{Z})$ is a $\mathcal{C}$-isomorphism for all $i < n$ and a $\mathcal{C}$-epimorphism for $i = n$.

- Proof sketch:
- First assume statement (1). This implies that $\pi_i(F_f) \in \mathcal{C}$ for $i < n$, so the mod-$\mathcal{C}$ Hurewicz theorem 6.26 implies that $H_i(F_f;\mathbb{Z}) \in \mathcal{C}$ for $i < n$. We may therefore apply the Fibration theorem 6.13 with $r = n$ and $s = 2$ to the fibration of pairs $(N_f, F_f) \to (Y,*)$ and conclude that $H_i(N_f,F_f) \to H_i(Y,*)$ is a $\mathcal{C}$-isomorphism in degrees $i \leqslant n+1$ and a $\mathcal{C}$-epimorphism in degree $i = n+2$. Then the mod-$\mathcal{C}$ 5-lemma applied to the map of long exact sequences of homology groups induced by $(N_f,F_f) \to (Y,*)$ implies statement (2).
- Now assume statement (2). By the mod-$\mathcal{C}$ Hurewicz theorem 6.26, it is enough to show that $H_i(F_f;\mathbb{Z}) \in \mathcal{C}$ for $i < n$. We prove this by induction on i. For $i \leqslant 1$ this is true since F_f is simply-connected. Let $2 \leqslant k < n$. By the inductive hypothesis, we may apply the Fibration theorem 6.13 to the fibration of pairs $(N_f,F_f) \to (Y,*)$ with $r = i$ and $s = 2$ to deduce that $H_j(N_f,F_f) \to H_j(Y,*)$ is a $\mathcal{C}$-isomorphism in degrees $j \leqslant i+1$ and a $\mathcal{C}$-epimorphism in degree $j = i+2$. Then the mod-$\mathcal{C}$ 5-lemma implies statement (1).

6.29 Definition (*Stiefel manifolds*): For $k \leqslant n$ define $V_{n,k}$ to be the orbit space $O(n)/O(n-k)$, in other words, the space of $O(n-k)$-cosets in the orthogonal group $O(n)$. This can also be thought of as the set of orthonormal k-tuples in $\mathbb{R}^n$. For example $V_{n,1} \cong S^{n-1}$. For any $k \leqslant \ell \leqslant n$ there is a fibre bundle $V_{n,\ell} \to V_{n,k}$ given by forgetting $\ell - k$ vectors from an ℓ-tuple. In particular, for any $1 \leqslant k \leqslant n$ we have a fibre bundle

$$V_{n,k} \longrightarrow S^{n-1}. \tag{17}$$

6.30 Fact: We will see in the next chapter of the lecture course that $V_{n,2}$ has a CW structure with exactly 4 cells, in dimensions $0, n-2, n-1, 2n-3$. Moreover, in the cellular chain complex of this CW-complex, all differentials are zero except when n is odd, in which case there is a single non-zero differential $\mathbb{Z} \cong C_{n-1}(V_{n,2}) \to C_{n-2}(V_{n,2}) \cong \mathbb{Z}$, which is multiplication by 2.

- Corollary: For $n \geqslant 3$,

$$H_i(V_{n,2};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \text{ or } 2n-3 \\ \mathbb{Z} & i = n-2 \text{ or } n-1, \text{ if } n \text{ is even} \\ \mathbb{Z}/2 & i = n-2, \text{ if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

(Note that $V_{2,2} = O(2) \cong S^1 \sqcup S^1$, so $H_i(V_{2,2};\mathbb{Z})$ is $\mathbb{Z}^2$ for $i = 0, 1$ and zero for $i \geqslant 2$.)

- Now we can apply everything we have proved so far in this chapter to prove the main results.

6.31 Theorem: Let $j > n$ and n be odd. Then $\pi_j(S^n)$ is finite.

6.32 Theorem: Let $j > n$ and n be even. Then $\pi_j(S^n)$ is finite, except for $\pi_{2n-1}(S^n)$, which is $\mathbb{Z} \oplus$ finite.

- Proof of Theorem 6.31:
- First of all, the result is clear for $n = 1$, so we may assume that $n \geqslant 3$. Let $f\colon S^n \to K(\mathbb{Z}, n)$ represent a generator of $\pi_n(K(\mathbb{Z}, n))$. By Theorem 6.24 and the universal coefficient theorem, f induces an isomorphism on $H_*(-;\mathbb{Q}) = H_*(-;\mathbb{Z}) \otimes \mathbb{Q}$. In other words, the induced map $H_*(f;\mathbb{Z})$ is a $\mathcal{T}$-isomorphism, for $\mathcal{T} = \{\text{torsion abelian groups}\}$. Since $\mathcal{T}$ is homology-closed (Lemma 6.25), we may apply the mod-$\mathcal{T}$ Whitehead theorem 6.28 to deduce that $\pi_i(f)$ is also a $\mathcal{T}$-isomorphism for all i. This implies that $\pi_j(S^n)$ is a torsion abelian group for $j > n$, since $\pi_j(K(\mathbb{Z}, n)) = 0$. But, by Corollary 6.27, $\pi_j(S^n)$ is also finitely generated, so it must be finite.
- Proof of Theorem 6.32:
- Write $n = 2k$ and $V = V_{2k+1,2}$ and consider the fibre bundle

$$p\colon V \longrightarrow S^{2k},$$

a special case of (17). The fibre over a point v is the space of unit vectors in $\mathbb{R}^{2k+1}$ that are orthogonal to v, which is homeomorphic to S^{2k-1}. There is a map $f\colon V \to S^{4k-1}$ inducing an isomorphism on $H_{4k-1}(-;\mathbb{Z})$, since V is a closed, orientable $(4k+1)$-dimensional manifold. (For example, choose a coordinate neighbourhood in V and collapse its complement to a point.) Since V has the rational homology of S^{4k-1} (by 6.30), the induced map $H_*(f;\mathbb{Q})$ is an isomorphism in all degrees. By the same argument as above, using the mod-$\mathcal{T}$ Whitehead theorem, we deduce that $\pi_i(f)$ is a $\mathcal{T}$-isomorphism for all i. By Corollary 6.27, the homotopy groups $\pi_i(V)$ are finitely generated for all i. Hence Theorem 6.31 and the fact that $\pi_i(f)$ is a $\mathcal{T}$-isomorphism (and the classification of finitely generated abelian groups) imply that $\pi_i(V)$ is finite for all i, except for $\pi_{4k-1}(V)$, which is $\mathbb{Z} \oplus$ finite. Using this fact, and applying Theorem 6.31 again to the homotopy groups of S^{2k-1}, the long exact sequence of homotopy groups of the fibre bundle p implies the result.

6.33 Aside on the *Hopf invariant*.

- There is a homomorphism (in fact isomorphism) $\pi_n(S^n) \to \mathbb{Z}$ given by sending $[f]$ to the integer $\deg(f)$ such that $f_*(\alpha) = \deg(f).\alpha$, where α is a fixed generator of $H_n(S^n)$.
- By Theorems 6.31 and 6.32, we know that the only other possible i for which there can be a non-trivial homomorphism $\pi_i(S^n) \to \mathbb{Z}$ is $i = 2n-1$ if n is even.

- A homomorphism $h\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$ may be defined as follows (for any $n \geqslant 2$, although it will necessarily be trivial if n is odd).
- Fix generators α of $H^n(S^n)$ and β of $H^{2n}(D^{2n}, S^{2n-1})$. Given an element $[f] \in \pi_{2n-1}(S^n)$ and a map $f\colon S^{2n-1} \to S^n$ representing it, construct a CW-complex $X(f)$ by attaching a $2n$-cell to S^n along f. The inclusion of the n-skeleton of $X(f)$ induces an isomorphism

$$\varphi\colon H^n(X(f)) \longrightarrow H^n(S^n)$$

and the characteristic map $(D^{2n}, S^{2n-1}) \to (X(f), S^n)$ of the $2n$-cell induces an isomorphism $H^{2n}(X(f), S^n) \to H^{2n}(D^{2n}, S^{2n-1})$, which may be composed with an isomorphism from the long exact sequence for $(X(f), S^n)$ to obtain an isomorphism

$$\psi\colon H^{2n}(X(f)) \longrightarrow H^{2n}(D^{2n}, S^{2n-1}).$$

We then define $h([f])$ to be the integer such that $\varphi^{-1}(\alpha) \cup \varphi^{-1}(\alpha) = h([f]).\psi^{-1}(\beta)$.
- Some facts that are easy to check from the definitions are:
  - This definition results in a well-defined homomorphism $h\colon \pi_{2n-1}(S^n) \to \mathbb{Z}$. This is the *Hopf map*, and $h([f])$ is the *Hopf invariant* of f.
  - For $n \geqslant 2$, given maps $S^{2n-1} \xrightarrow{a} S^{2n-1} \xrightarrow{b} S^n \xrightarrow{c} S^n$, we have the following identities:

$$h([b \circ a]) = h([b]).\deg(a) \qquad\qquad h([c \circ b]) = h([b]).\deg(c)^2.$$

- Theorem (Hopf): The image of h contains $2\mathbb{Z}$.
- Theorem (Adams): The image of h is $\mathbb{Z}$ (i.e. h is surjective) if and only if $n = 2, 4$ or 8.

22. Wednesday 9 January

- Recall the *Whitehead tower* of a path-connected space (see 4.57(b)). Namely, for any path-connected space Y there is a sequence (tower) of principal fibrations

$$\cdots \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_2 \longrightarrow Y_1 \longrightarrow Y,$$

where Y_n is n-connected, the map $Y_n \to Y_{n-1}$ is *n-coconnected* (induces isomorphisms on π_i for $i > n$ and an injection on π_n) and its fibre is $K(\pi_n(Y), n-1)$. This sequence is unique up to homotopy-equivalence of diagrams. In particular, the principal fibration $Y_1 \to Y$ is the universal cover of Y (when Y admits a universal cover, i.e., when Y is locally path-connected and semi-locally simply-connected). The composite map $Y_n \to Y$ is analogously called the *n-connected cover* of Y. In the proof of the mod-$\mathcal{C}$ Hurewicz theorem (Theorem 6.26), we will need to use the 2-connected cover of a space.
- We now restate the mod-$\mathcal{C}$ Hurewicz theorem. For a path-connected space X we have the following statements (for $n \geqslant 2$):

$$\begin{aligned} \Pi(n) &= \pi_i(X) \in \mathcal{C} \text{ for all } i < n \\ \mathrm{H}(n) &= H_i(X;\mathbb{Z}) \in \mathcal{C} \text{ for all } i < n \\ \mathrm{I}(n) &= \text{The Hurewicz map } \pi_i(X) \to H_i(X;\mathbb{Z}) \text{ is a } \mathcal{C}\text{-isomorphism for all } i \leqslant n \end{aligned}$$

Fix $n \geqslant 2$, a homology-closed Serre class of $\mathbb{Z}$-modules $\mathcal{C}$ and a 1-connected space X. Assume:
(i) either $\mathcal{C}$ is saturated
(ii) or $X \in \mathcal{G}(n, \mathbb{Z})$, i.e., $H_i(X;\mathbb{Z})$ is finitely generated for all $i < n$.

6.34 Theorem (also Theorem 6.26): Under these conditions, $\Pi(n) \Leftrightarrow \mathrm{H}(n) \Rightarrow \mathrm{I}(n)$.
- Proof:
The path-fibration $f\colon (PX, \Omega X) \to (X, *)$, the long exact sequences of the pair $(PX, \Omega X)$ and the Hurewicz homomorphisms induce a commutative diagram

$$\begin{tikzcd} \pi_n(X) \arrow[d, "h"] & \pi_n(PX,\Omega X) \arrow[l, "\pi_n(f)"', "\cong"] \arrow[r, "\partial", "\cong"'] \arrow[d] & \pi_{n-1}(\Omega X) \arrow[d, "h_\Omega"] \\ H_n(X,*) & H_n(PX,\Omega X) \arrow[l, "H_n(f)"] \arrow[r, "\cong", "\partial"'] & H_{n-1}(\Omega X), \end{tikzcd} \tag{18}$$

where the two boundary maps ∂ are isomorphisms because PX is contractible, and the map $\pi_n(f)\colon \pi_n(PX, \Omega X) \to \pi_n(X)$ is an isomorphism because f is a Serre fibration (*cf.* the proof of Proposition 2.62). We abbreviate $H_*(-;\mathbb{Z})$ to $H_*(-)$ in this proof.

- The proof is by induction on n. The statements $\Pi(2)$ and $\mathrm{H}(2)$ are always true, since X is 1-connected, and the statement $\mathrm{I}(2)$ is also always true, by the classical Hurewicz theorem (Theorem 1.30). So the base case $n = 2$ holds.
- Let $n \geqslant 3$. We first show the easy implications, which are $\Pi(n) \Leftrightarrow \mathrm{H}(n)$.
  Statement $\mathrm{H}(n)$ implies $\mathrm{H}(n-1)$, and therefore $\mathrm{I}(n-1)$ by the inductive hypothesis. This is the statement that $\pi_i(X) \to H_i(X)$ is a $\mathcal{C}$-isomorphism for all $i \leqslant n-1$. By $\mathrm{H}(n)$ we know that $H_i(X) \in \mathcal{C}$ in this range of degrees, and therefore so is $\pi_i(X)$. This is $\Pi(n)$.
  The opposite direction is identical: statement $\Pi(n)$ implies $\Pi(n-1)$, which implies $\mathrm{I}(n-1)$ by inductive hypothesis, and then statements $\Pi(n)$ and $\mathrm{I}(n-1)$ together imply $\mathrm{H}(n)$.
- We will prove the remaining implication $\Pi(n) \Rightarrow \mathrm{I}(n)$ in two steps.
  Claim$_1$: Assuming that Theorem 6.34 holds for smaller values of n, and assuming that X is 2-connected, we have the implication $\Pi(n) \Rightarrow \mathrm{I}(n)$.
  Claim$_2$: Assuming that Theorem 6.34 holds for smaller values of n, and assuming that *Claim*$_1$ is true, we have the implication $\Pi(n) \Rightarrow \mathrm{I}(n)$.
  Once we have proven *Claim*$_1$ and *Claim*$_2$, the induction will be complete.
- Proof of *Claim*$_1$:
  The idea is to apply the inductive hypothesis to ΩX, so we first have to check that the hypotheses of the theorem apply to it. It is 1-connected by our extra assumption that X is 2-connected. If $\mathcal{C}$ is saturated, then we have nothing further to check. If, on the other hand, we have made assumption (ii) that $X \in \mathcal{G}(n,\mathbb{Z})$, then we must check that $\Omega X \in \mathcal{G}(n-1,\mathbb{Z})$, in order to apply the inductive hypothesis to it. This follows by Corollary 6.15(b).
  Thus, by induction, we have the implication $\Pi(n-1) \Rightarrow \mathrm{H}(n-1)$ and $\mathrm{I}(n-1)$ for ΩX.
  For $i < n-1$, we know $\pi_i(\Omega X) \cong \pi_{i+1}(X) \in \mathcal{C}$, by $\Pi(n)$ for X, so $\Pi(n-1)$ holds for ΩX.
  Now $\mathrm{I}(n-1)$ for ΩX implies that the map h_Ω in diagram (18) is a $\mathcal{C}$-isomorphism.
  Statement $\mathrm{H}(n-1)$ for ΩX implies that we may apply the Fibration theorem 6.13 to the fibration of pairs $(PX, \Omega X) \to (X, *)$ with $s = 2$ and $r = n-1$. (If $\mathcal{C}$ is saturated, use part (a) of the theorem, if we have assumed that $X \in \mathcal{G}(n,\mathbb{Z})$, use part (b) with $d = n-1$.) The Fibration theorem implies that the map $H_n(f)$ in diagram (18) is a $\mathcal{C}$-isomorphism.
  Thus we have shown that $h\colon \pi_i(X) \to H_i(X)$ is a $\mathcal{C}$-isomorphism for $i = n$. For $i \leqslant n-1$ it is also a $\mathcal{C}$-isomorphism (since $\Pi(n) \Rightarrow \Pi(n-1) \Rightarrow \mathrm{I}(n-1)$ by inductive hypothesis), so we have shown $\mathrm{I}(n)$ for X. This completes the proof of *Claim*$_1$.
- Proof of *Claim*$_2$:
  Consider the 2-connected cover of X, a fibration
  $$X_2 \longrightarrow X,$$
  whose fibre is $K(A,1)$, where $A = \pi_2(X)$. The space X_2 is 2-connected and the map $X_2 \to X$ induces isomorphisms on π_i for $i \geqslant 3$. This fibration and the Hurewicz homomorphisms give us a commutative diagram
  $$\begin{tikzcd} \pi_n(X_2) \arrow[r, "h_2"] \arrow[d, "\cong"'] & H_n(X_2) \arrow[d] \\ \pi_n(X) \arrow[r, "h"'] & H_n(X), \end{tikzcd} \tag{19}$$
  where the left-hand vertical map is an isomorphism since $n \geqslant 3$.
  The idea is to apply *Claim*$_1$ to X_2, so we first have to check that the hypotheses of Theorem 6.34 apply to X_2. It is certainly 1-connected. If $\mathcal{C}$ is saturated, then we have nothing further to check. If, on the other hand, we have instead made assumption (ii) that $X \in \mathcal{G}(n,\mathbb{Z})$, then we must check that $X_2 \in \mathcal{G}(n,\mathbb{Z})$. Part (E) of Corollary 6.14 with $\mathcal{C} = \mathcal{G}$, applied to the fibration $X_2 \to X$, says that
  $$X \text{ and } K(A,1) \in \mathcal{G}(n,\mathbb{Z}) \implies X_2 \in \mathcal{G}(n,\mathbb{Z}).$$

So we need to know that $K(A,1) \in \mathcal{G}(n,\mathbb{Z})$. To see this, note that $A = \pi_2(X) \cong H_2(X)$ by the classical Hurewicz theorem (1.30). We know that $X \in \mathcal{G}(n,\mathbb{Z})$ and $n \geqslant 3$, so $A \in \mathcal{G}$. Proposition 6.16 then implies that $K(A,1) \in \mathcal{G}(\infty,\mathbb{Z})$.
Thus the hypotheses of Theorem 6.34 apply to X_2, so by *Claim*$_1$ we have the implication $\Pi(n) \Rightarrow \mathrm{I}(n)$ for X_2.
Statement $\mathrm{I}(n)$ for X_2 implies that the map h_2 in diagram (19) is a $\mathcal{C}$-isomorphism.
Claim$_3$: Assuming that Theorem 6.34 holds for smaller values of n, and assuming statement $\Pi(n)$ for X, the map $H_i(X_2) \to H_i(X)$ is a $\mathcal{C}$-isomorphism for $2 \leqslant i \leqslant n$.
We will prove this in a moment. First we use it to complete the proof of *Claim*$_2$.
Diagram (19), the fact that h_2 is a $\mathcal{C}$-isomorphism and *Claim*$_3$ imply that the map $h\colon \pi_i(X) \to H_i(X)$ is a $\mathcal{C}$-isomorphism for $i = n$. For $i \leqslant n-1$ it is also a $\mathcal{C}$-isomorphism (since we have implications $\Pi(n) \Rightarrow \Pi(n-1) \Rightarrow \mathrm{I}(n-1)$ by inductive hypothesis), so we have shown $\mathrm{I}(n)$ for X. This completes the proof of *Claim*$_2$, assuming *Claim*$_3$.

- Proof of *Claim*$_3$:[9]
  Since $n \geqslant 3$, statement $\Pi(n)$ for X implies that $A = \pi_2(X) \in \mathcal{C}$. We have assumed that $\mathcal{C}$ is homology-closed, so $K(A,1) \in \mathcal{C}(\infty,\mathbb{Z})$. Thus we may apply the Fibration theorem 6.13 to the fibration of pairs
  $$(X_2, K(A,1)) \longrightarrow (X,*) \tag{20}$$
  with $s = 2$ and $r = \infty$. If $\mathcal{C}$ is saturated, part (a) of that theorem implies that the map on relative homology induced by (20) is a $\mathcal{C}$-isomorphism in all degrees. If instead we have assumed that $X \in \mathcal{G}(n,\mathbb{Z})$, then part (b) of that theorem (with $d = n-1$) implies that the map on relative homology induced by (20) is a $\mathcal{C}$-isomorphism in degrees $\leqslant n$.
  Let $2 \leqslant i \leqslant n$. In the diagram
  $$\begin{tikzcd} H_i(K(A,1)) \arrow[r] \arrow[d] & H_i(X_2) \arrow[r, "(\diamond)"] \arrow[d] & H_i(X_2, K(A,1)) \arrow[r] \arrow[d, "(\star)"] & H_{i-1}(K(A,1)) \arrow[d] \\ H_i(*) \arrow[r] & H_i(X) \arrow[r] & H_i(X,*) \arrow[r] & H_{i-1}(*) \end{tikzcd} \tag{21}$$
  the rows are exact and the map $(\star)$ is a $\mathcal{C}$-isomorphism, by above. Since $K(A,1) \in \mathcal{C}(\infty,\mathbb{Z})$ and by exactness, the map $(\diamond)$ is also a $\mathcal{C}$-isomorphism. The map $H_i(X) \to H_i(X,*)$ is of course an (ordinary) isomorphism, so we conclude that $H_i(X_2) \to H_i(X)$ is a $\mathcal{C}$-isomorphism. This completes the proof of *Claim*$_3$, and of Theorem 6.34.

- We will finish this chapter with a brief remark about rational homology theories.
- Recall that cw_* denotes the 2-category of zero-based CW-complexes, based maps and based homotopies, and a reduced homology theory is a sequence of functors $h_n\colon \mathrm{Ho}(\mathrm{cw}_*) \to \mathrm{Ab}$ from the associated homotopy category to the category of abelian groups, indexed by $n \in \mathbb{Z}$, satisfying the wedge and exactness axioms, together with a sequence of natural isomorphisms $h_n \to h_{n+1} \circ \Sigma$, where Σ is the endofunctor of $\mathrm{Ho}(\mathrm{cw}_*)$ given by reduced suspension.

6.35 Examples:
  (i) $h_n(X) = \widetilde{H}_n(X;\mathbb{Z})$. The *coefficients* $h_n(S^0)$ of this homology theory are $\mathbb{Z}$ in degree $n = 0$ and 0 in degrees $n \neq 0$.
  (ii) $h_n(X) = \pi_n^{\mathrm{st}}(X) = \mathrm{colim}_{k\to\infty}(\pi_{n+k}(\Sigma^k X))$. The coefficients of this homology theory are the stable homotopy groups of spheres $\pi_n^{\mathrm{st}}(S^0)$. By Theorems 6.31 and 6.32, these are $\mathbb{Z}$ in degree $n = 0$ and *finite* in degrees $n \neq 0$.

- The Hurewicz homomorphisms make the following square commute:
  $$\begin{tikzcd} \pi_{n+k}(\Sigma^k X) \arrow[d, "h"'] & \pi_{n+k+1}(C\Sigma^k X, \Sigma^k X) \arrow[l, "\cong"'] \arrow[r] & \pi_{n+k+1}(\Sigma^{k+1}X) \arrow[d, "h"] \\ \widetilde{H}_{n+k}(\Sigma^k X;\mathbb{Z}) \arrow[rr, "\cong", "\Sigma_*"'] & & \widetilde{H}_{n+k+1}(\Sigma^{k+1}X;\mathbb{Z}), \end{tikzcd}$$

[9] Don't worry, there is no *Claim*$_4$!

so they induce a *stable Hurewicz homomorphism*

$$h_n^{\mathrm{st}} \colon \pi_n^{\mathrm{st}}(X) \longrightarrow \widetilde{H}_n(X;\mathbb{Z}).$$

Moreover, this is a morphism of homology theories (it is natural in X and it commutes with the natural isomorphisms $h_n \to h_{n+1} \circ \Sigma$).

6.36 Definition: A reduced homology theory h is *rational* if each of the functors $h_n \colon \mathrm{Ho}(\mathrm{cw}_*) \to \mathrm{Ab}$ takes values in the subcategory $\mathrm{Vect}_{\mathbb{Q}} \leqslant \mathrm{Ab}$ of rational vector spaces.

6.37 Theorem: Let h be any rational homology theory. Then there is an isomorphism of homology theories:

$$h_n(X) \;\cong\; \bigoplus_{p+q=n} \widetilde{H}_p(X; h_q(S^0)).$$

- Sketch proof: There is a homomorphism

$$\pi_p^{\mathrm{st}}(X) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(h_q(S^0), h_{p+q}(X))$$

defined as follows. Let $f \colon S^{p+k} \to \Sigma^k X$ represent an element of π_p^{st}. Send this to the map

$$h_q(S^0) \cong h_{q+p+k}(S^{p+k}) \xrightarrow{f_*} h_{q+p+k}(\Sigma^k X) \cong h_{q+p}(X), \tag{22}$$

where the isomorphisms on each side are suspension isomorphisms. The adjoint of this is a map $\pi_p^{\mathrm{st}}(X) \otimes_{\mathbb{Z}} h_q(S^0) \to h_{p+q}(X)$, and summing these over all $p+q=n$ gives us a map

$$\bigoplus_{p+q=n} \pi_p^{\mathrm{st}}(X) \otimes_{\mathbb{Z}} h_q(S^0) \longrightarrow h_n(X). \tag{23}$$

This is in fact a morphism of homology theories. Using the stable Hurewicz homomorphisms, we have another morphism of homology theories

$$\bigoplus_{p+q=n} \pi_p^{\mathrm{st}}(X) \otimes_{\mathbb{Z}} h_q(S^0) \longrightarrow \bigoplus_{p+q=n} \widetilde{H}_p(X;\mathbb{Z}) \otimes_{\mathbb{Z}} h_q(S^0) \;\cong\; \bigoplus_{p+q=n} \widetilde{H}_p(X; h_q(S^0)), \tag{24}$$

where the isomorphism on the right is due to the universal coefficient theorem (there are no Tor terms involved, because $h_q(S^0)$ is a rational vector space). We just need to show that (23) and (24) are isomorphisms. By Proposition 4.45, it suffices to check this when $X = S^0$.

- For (23): Since $\pi_p^{\mathrm{st}}(S^0)$ is finite for $p \neq 0$ and $h_q(S^0)$ is a rational vector space, the left-hand side of (23) (when $X = S^0$) is $\pi_0^{\mathrm{st}}(S^0) \otimes_{\mathbb{Z}} h_n(S^0) \cong h_n(S^0)$. Unwinding the definitions, the map (23) is identified with the map (22) when $p=0$, $q=n$, $X=S^0$ and $f = \mathrm{id} \colon S^0 \to S^0$, so in particular it is an isomorphism.
- For (24): As above, the left-hand side of (24) (when $X = S^0$) is $\pi_0^{\mathrm{st}}(S^0) \otimes_{\mathbb{Z}} h_n(S^0)$. By the same argument, the right-hand side is $\widetilde{H}_0(S^0;\mathbb{Z}) \otimes_{\mathbb{Z}} h_n(S^0)$. The map (24) is identified with $h_0^{\mathrm{st}} \otimes \mathrm{id}$, where h_0^{st} is the stable Hurewicz homomorphism for S^0 in dimension 0. Since the sequence $\pi_0(S^0) \to \pi_1(S^1) \to \pi_2(S^2) \to \cdots$ stabilises already at $\pi_1(S^1)$, we may identify this with $h_1 \otimes \mathrm{id}$, where h_1 is the (unstable) Hurewicz homomorphism $\pi_1(S^1) \to H_1(S^1;\mathbb{Z})$, which is an isomorphism.

23. Monday 14 January

7. Principal bundles, vector bundles, classifying spaces

7.1 Definition of *fibre bundle* / *locally trivial map*.

- Examples: product bundles, covering spaces, the Möbius band, more generally the projection $([0,1] \times \mathbb{R})/\!\sim\; \to S^1$ where $\sim$ is generated by $(0,t) \sim (1,\varphi(t))$ for a fixed homeomorphism $\varphi \colon \mathbb{R} \to \mathbb{R}$.

7.2 Definition of *coordinate bundle* with base space B, structure group G and fibre F. (A locally trivial map equipped with an atlas of local trivialisations (or *local coordinates*) such that the change-of-coordinates functions $U_i \cap U_j \to \mathrm{Homeo}(F)$ factor continuously through the action $G \to \mathrm{Homeo}(F)$.)

7.3 Definition of *coordinate cocycle*.
7.4 Observation: there is a well-defined function

$$\mathrm{Cocyc}\colon \{\text{coordinate bundles for } B, F, G\} \longrightarrow \{\text{coordinate cocycles for } B, G\}.$$

7.5 Definition: (a) Two coordinate bundles are *compatible* if they have the same map, and the union of the two atlases for this map is again an atlas. (b) A *fibre bundle* with base space B, structure group G, and fibre F is an equivalence class of coordinate bundles with respect to the equivalence relation of compatibility.
7.6 Remark: If $G = \mathrm{Homeo}(F)$, then any locally trivial map has exactly one equivalence class (with respect to compatibility) of atlases, since any two atlases are compatible. So *fibre bundle with fibre F and structure group Homeo(F)* recovers the notion of *fibre bundle* (with unspecified structure group).
7.7 Remark: if $p\colon E \to B$ is a fibre bundle, then p is an open quotient map.
7.8 Definition of *bundle map*.
7.9 Remark: there is a well-defined function

$$\begin{aligned}&\{\text{bundle maps from } E_1 \to B_1 \text{ to } E_2 \to B_2\}\\ &\longrightarrow \{\text{abstract mapping transformations from } B_1 \text{ to } B_2 \text{ with values in } G\}\end{aligned}$$

7.10 Lemma: this is a bijection.
7.11 Corollary: a bundle map is invertible if and only if the map of base spaces is invertible.
7.12 Definition:
- (a) Note that, by definition, two coordinate bundles are compatible if and only if the identity map between them is a bundle map.
- (b) Two coordinate bundles are *equivalent* if there is a bundle map between them whose map of base spaces is the identity.
- (c) Note that compatible bundles are equivalent, and that "equivalent" is indeed an equivalence relation on coordinate bundles (by Corollary 7.11).
- (d) Two fibre bundles are *equivalent* if they have representative coordinate bundles that are equivalent.

7.13 Corollary: two coordinate bundles are equivalent if and only if a certain condition $(*)$ is satisfied by their associated coordinate cocycles.
7.14 Definition: two coordinate cocycles are *equivalent* if and only if $(*)$ holds.
7.15 Observation: if we quotient out by the equivalence relations on the source and target of the map Cocyc from (7.4), we obtain a well-defined and injective function, which we denote by

$$\overline{\mathrm{Cocyc}}\colon \mathrm{Bun}(B, F, G) \longrightarrow \mathrm{Cocyc}(B, G).$$

7.16 Theorem: This is a bijection: $\mathrm{Bun}(B, F, G) \cong \mathrm{Cocyc}(B, G)$.
7.17 Remark: This makes "change of fibres" easy to define: if F, F' are two faithful left G-spaces, we have canonical bijections

$$\mathrm{Bun}(B, F, G) \cong \mathrm{Cocyc}(B, G) \cong \mathrm{Bun}(B, F', G).$$

In particular, this means that the classification of $\mathrm{Bun}(B, F, G)$ depends only on B and G, not on F. So it will suffice to classify $\mathrm{Bun}(B, F, G)$ when $F = G$ ...
7.18 Examples:
- (a) Principal G-bundles (where $F = G$ acting on itself by left-multiplication). When G is discrete these are regular covering spaces.
- (b) Real vector bundles (where $F = \mathbb{R}^n$ and $G = GL_n(\mathbb{R})$ for some $n \in \mathbb{N}$). For example tangent bundles of smooth manifolds or normal bundles of immersions between smooth manifolds.

24. Wednesday 16 January

- Proof of Theorem 7.16.

7.19 Definition of *principal G-bundle.* A right G-space E and a G-invariant map $p\colon E \to B$ that is locally G*-equivariantly* trivial.
7.20 Remark: this definition is equivalent to the special case of a fibre bundle over B with structure group G and fibre $F = G$, considered as a left G-space by left-multiplication.
7.21 Remark: if $(E \curvearrowleft G, p\colon E \to B)$ is a principal G-bundle, then p is homeomorphic (as an object of E/Top) to the quotient map $E \to E/G$. Hence principal G-bundles are just (free) right G-spaces with a certain property.
7.22 Remark: The *change of fibre* of (7.17) may be described as follows, in the case where we start with a principal G-bundle, and we want to change the fibres to F. Let $E \to B$ be a principal G-bundle and consider the projection $E \times F \to E$. This is G-equivariant, so we may quotient out by the action of G to obtain a fibre bundle $E \times_G F \to B$ with structure group G and fibre F.
7.23 Definition: The *translation map* t_E of a free right G-space E. The action is *weakly proper* if t_E is continuous.
7.24 Lemma: If E is a free right G-space such that the projection $E \to E/G$ is a locally trivial map, then the action of G on E is weakly proper. In particular, any principal G-bundle is weakly proper.
7.25 Let E be a weakly proper free G-space. Then $p\colon E \to E/G$ is *trivial* (isomorphic to the projection $(E/G) \times G \to E/G$) if and only if there exists a section of p.
7.26 Definition: When G is discrete, principal G-bundles are also known as *principal G-coverings*.
7.27 Definition of *properly discontinuous* group actions. Lemma: Let E be a free G-space, for a discrete group G. Then $E \to E/G$ is a principal G-bundle if and only if the action of G on E is properly discontinuous.
7.28 Remark: These statements are also equivalent to the statement that the action of G on E is weakly proper. So:

$$\{\text{principal } G\text{-coverings}\} \cong \{\text{free, weakly proper } G\text{-spaces}\}.$$

In general, for non-discrete G, not all free, weakly proper G-spaces are principal G-bundles.
7.29 Examples: If E is a topological group and $G \leqslant E$ is a closed, discrete subgroup, then the action of G on E is free and weakly proper, and therefore $E \to E/G$ is a principal G-covering. In particular,
- $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$,
- $\mathbb{C} \to \mathbb{C}/\mathbb{Z} \cong S^1 \times \mathbb{R}$,
- $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$

are all principal $\mathbb{Z}$-coverings.
7.30 Lemma: Let U be a right G-space. Then $U \to U/G$ is trivial (isomorphic to the projection $(U/G) \times G \to U/G$) if and only if there exists a G-equivariant map $U \to G$.
7.31 Corollary: Let E be a right G-space. Then $E \to E/G$ is a principal G-bundle if and only if there is an open cover $\{U_i\}$ of E by G-invariant subspaces $U_i \subseteq E$, and G-equivariant maps $U_i \to G$.

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7.32 Definition of a *bundle map* between principal G-bundles: just a G-equivariant map between the total spaces.
7.33 Remark: this agrees with Definition 7.8 in the special case where $F = G$. Hence, by Corollary 7.11, a G-equivariant map $E_1 \to E_2$ between principal G-bundles is invertible if and only if the induced map $E_1/G \to E_2/G$ is invertible.
7.34 Remark about pullbacks and bundle maps:
- The pullback of a fibre bundle is again a fibre bundle (with the same fibre and the same structure group).
- If we have a commutative square in which both vertical maps are fibre bundles (with the same fibre and the same structure group), then the square is a bundle map if and only if it is a pullback square.

7.35 Lemma: Suppose that $f\colon E_1 \to E_2$ is a G-equivariant map of right G-spaces, and $E_2 \to E_2/G$ is a principal G-bundle. Then the square

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d] & E_2 \arrow[d] \\ E_1/G \arrow[r, "f/G"'] & E_2/G \end{tikzcd}$$

is a pullback square.

- Proof: By Remark 7.34, it will be enough to show that $E_1 \to E_1/G$ is a principal G-bundle; this can be shown using the characterisation in Corollary 7.31.

7.36 Corollary: Suppose that E is a topological group and $G \leqslant E$ is a subgroup. The quotient $p\colon E \to E/G$ is a principal G-bundle if and only if there exists a local section of p near the coset $1_E.G \in E/G$.

7.37 Proposition: Let E be a weakly proper, free right G-space and let F be a left G-space. Consider the commutative square:

$$\begin{tikzcd} E \times F \arrow[r, "\mathrm{pr}_1"] \arrow[d] & E \arrow[d] \\ E \times_G F \arrow[r, "q"'] & E/G \end{tikzcd}$$

There is a one-to-one correspondence between G-equivariant maps $E \to F$ (meaning maps f such that $f(xg) = g^{-1}f(x)$ for all $g \in G$ and $x \in E$) and sections of q. Moreover, this correspondence may be parametrised by $[0,1]$, in the sense that G-equivariant homotopies of maps $E \to F$ correspond to homotopies of sections of q.

7.38 Definitions:
- The *support* of a continuous function $X \to [0, \infty)$.
- A *locally finite* collection of subsets of X. A *locally finite* collection of functions on X.
- A *partition of unity*.
- A *partition of unity subordinate to* a given open cover of X. Numerable open covers of X.

7.39 Recall that we gave a different definition of the property of being "numerable" for an open cover in Definition 2.29. From now on we will call this property *strongly numerable*.

7.40 Observation: if $\mathcal{U}$ is a numerable open cover of X, then there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is strongly numerable. Conversely, we have:

7.41 Lemma: if $\mathcal{U}$ is strongly numerable, then it is numerable.

7.42 Definition of *numerable bundle* (the base space admits a numerable open cover that locally trivialises the bundle).
Corollary: every numerable bundle is a Hurewicz fibration.
(This follows from 7.40 and 2.37(b).)

7.43 Theorem: If $\mathcal{U}$ is a locally finite open cover of X, and X is a normal space, then $\mathcal{U}$ is numerable.

7.44 Lemma: If $\mathcal{V}$ is a refinement of $\mathcal{U}$, and $\mathcal{V}$ is numerable, then $\mathcal{U}$ is also numerable.

7.45 Corollary: If X is paracompact and Hausdorff, then every open cover of X is numerable.

7.46 Remark: For any space B, we will classify the set of *numerable* principal G-bundles over B up to isomorphism. If B is paracompact and Hausdorff, this will be the set of *all* principal G-bundles over B up to isomorphism.

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7.47 Proposition: Let $\mathcal{U}$ be a numerable open cover of X. Then there is a *countable* numerable open cover $\mathcal{V}$ of X such that each $V \in \mathcal{V}$ is of the form

$$V = \coprod_{i \in I_V} V_i,$$

where each V_i is an open subset of X that is contained in some element of $\mathcal{U}$ (in other words, the open cover $\{V_i \mid i \in I_V, V \in \mathcal{V}\}$ is a refinement of $\mathcal{U}$).

7.48 Corollary: if $p\colon E \to B$ is a numerable bundle, then there is a *countable* numerable open cover of B that locally trivialises p.

7.49 Theorem: Let $\mathcal{U}$ be a numerable open cover of $B \times [0,1]$. Then there is a numerable open cover $\mathcal{V}$ of B and a function $\epsilon\colon \mathcal{V} \longrightarrow (0,\infty)$ such that the covering

$$\big\{V \times [s,t] \mid V \in \mathcal{V},\ 0 < t-s < \epsilon(V)\big\}$$

of $B \times [0,1]$ is a refinement of $\mathcal{U}$.

7.50 Theorem: Let $p\colon E \to B \times [0,1]$ be a numerable bundle with fibre F and structure group G, and let $r\colon B\times[0,1] \to B\times[0,1]$ be the self-map defined by $r(b,t) = (b,1)$. Then the pullback of p along r is p. More precisely, there is a bundle map (equivalently, pullback square) of the form

$$\begin{tikzcd} E \arrow[r, "R"] \arrow[d, "p"'] & E \arrow[d, "p"] \\ B \times [0,1] \arrow[r, "r"'] & B \times [0,1], \end{tikzcd}$$

and the restriction of R to $p^{-1}(B \times \{1\})$ is the identity.

(The proof involves writing r as a countably infinite (but locally finite) composition $\bigcirc_{i=0}^{\infty} r_i$ of self-maps of $B \times [0,1]$, lifting each r_i to a self-map R_i of E and then taking the infinite composition $R = \bigcirc_{i=0}^{\infty} R_i$.)

7.51 Remark: We used Theorem 7.49 and then Proposition 7.47 to find a countable numerable open cover of B, which we then used to index the composition of self-maps of $B \times [0,1]$. In particular, Proposition 7.47 allowed us to index the composition by $\mathbb{N}$. An alternative proof, avoiding the use of Proposition 7.47, uses the open cover of B provided by Theorem 7.49 directly. By the well-ordering theorem (equivalent to the axiom of choice), we can choose a bijection between that open cover and a (possibly uncountable) ordinal number α. Then we decompose r as a transfinite composition of self-maps r_i for $i \in \alpha$, and the rest of the proof works as before. [This does not obviate the need for Proposition 7.47, however; we will need it later, for Lemma 7.65, in the construction of a universal principal G-bundle.]

7.52 Theorem: Let $p\colon E \to B \times [0,1]$ be a numerable bundle with fibre F and structure group G. Then the restrictions

$$\begin{aligned} p_0\colon E_0 = p^{-1}(B \times \{0\}) &\longrightarrow B \times \{0\} \cong B \\ p_1\colon E_1 = p^{-1}(B \times \{1\}) &\longrightarrow B \times \{1\} \cong B \end{aligned}$$

are isomorphic bundles over B.

7.53 Corollary (*The homotopy theorem*): Let $p\colon E \to B \times [0,1]$ be a numerable bundle with fibre F and structure group G and let $f, g\colon A \to B$ be homotopic maps. Then the pullbacks $f^*(p)\colon f^*(E) \to A$ and $g^*(p)\colon g^*(E) \to A$ are isomorphic bundles over A.

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7.54 Definition: Let NPrin_G be the 2-category whose objects are numerable principal G-bundles, whose 1-morphisms are bundle maps (equivalently, G-equivariant maps between total spaces) and whose 2-morphisms are bundle homotopies (equivalently, G-equivariant homotopies). Identifying all 1-morphisms that are connected by 2-morphisms, we obtain the homotopy 1-category $\mathrm{Ho}(\mathrm{NPrin}_G)$, consisting of numerable principal G-bundles and bundle-homotopy classes of bundle maps (equivalently, G-equivariant homotopy classes of G-equivariant maps). A numerable principal G-bundle is called *universal* if it is a terminal object of $\mathrm{Ho}(\mathrm{NPrin}_G)$.

7.55 Theorem: For any topological group G, there exists a universal principal G-bundle.
(This will be proved later, as Lemmas 7.64–66.)

7.56 Remark: Universal principal G-bundles are unique up to bundle-homotopy equivalence. In particular, base spaces of universal principal G-bundles are unique up to homotopy equivalence.

7.57 Definition: The *classifying space of* G, denoted BG, is the base space of a universal principal G-bundle.

Notation: Let us write $p_G\colon EG \to BG$ for the universal principal G-bundle, and $\mathrm{PBun}_N(B,G)$ for the set of isomorphism classes of numerable principal G-bundles over B.

7.58 Corollary (*Classification theorem, I*): For any space B and topological group G, there is a bijection $[B,BG] \cong \mathrm{PBun}_N(B,G)$ given by sending a homotopy class $[f]$ to the pullback bundle $f^*(p_G)\colon f^*(EG) \to B$.

7.59 Corollary (*Classification theorem, II*): For any space B, topological group G and faithful left G-space F, there is a bijection $[B,BG] \cong \mathrm{Bun}_N(B,F,G)$ given by sending a homotopy class $[f]$ to the pullback bundle $f^*(p_G \times_G \mathrm{id}_F)\colon f^*(EG \times_G F) \to B$.

- To prove Theorem 7.55 (and hence the classification theorems), we first need another fact about partitions of unity.

7.60 Lemma: Let $\{f_i\colon X \to [0,\infty) \mid i \in I\}$ be a collection of continuous functions such that, for each $x \in X$, the set $\{i \in I \mid f_i(x) \neq 0\}$ is finite and $\sum_{i\in I} f_i(x) = 1$. (This is a *generalised partition of unity.*) Then the open cover

$$\left\{f_i^{-1}((0,\infty)) \mid i \in I\right\}$$

of X is numerable.

7.61 Definition: Let $\{X_i \mid i \in I\}$ be a collection of spaces. Their *join*

$$X = \mathop{\bigstar}_{i\in I} X_i$$

is the following space:

- As a set, it is a quotient of a subset of $[0,1]^I \times \prod_{i\in I} X_i$. First we take the subset of collections $\{t_i x_i \mid i \in I\}$ with $t_i \in [0,1]$ and $x_i \in X_i$ such that only finitely many of the t_i are non-zero, and $\sum_{i\in I} t_i = 1$. Then we identify $0x_i$ with $0x_i'$ for any $x_i, x_i' \in X_i$. More precisely, two collections $\{t_i x_i \mid i \in I\}$ and $\{u_i y_i \mid i \in I\}$ are identified if and only if $t_i = u_i$ for all $i \in I$ and $x_i = y_i$ for all $i \in I$ such that $t_i = u_i \neq 0$.
- There are well-defined functions

$$\begin{aligned} p_j\colon X &\longrightarrow [0,1] & \{t_i x_i\} &\mapsto t_j \\ q_j\colon p_j^{-1}((0,1]) &\longrightarrow X_j & \{t_i x_i\} &\mapsto x_j. \end{aligned}$$

- Then X is given the smallest topology such that all p_j and q_j are continuous. In other words, it is generated by the subsets $p_j^{-1}(A)$ and $q_j^{-1}(B)$ where A belongs to a chosen basis for the topology of $[0,1]$ and B belongs to a chosen basis for the topology of X_j.

7.62 Remarks:

(a) A map into $\mathop{\bigstar}_{i\in I} X_i$ is continuous if and only if its composition with each p_j and each q_j is continuous.

(b) Using this, one can check that, if each X_i is a right G-space, then $\mathop{\bigstar}_{i\in I} X_i$ is also naturally a right G-space.

7.63 Definition: For a topological group G, the *Milnor G-space* is

$$G^{\star\infty} = \mathop{\bigstar}_{i\in\mathbb{N}} X_i, \qquad \text{where } X_i = G \text{ for all } i \in \mathbb{N}.$$

This is also called the (countably) *infinite join power* of G.

7.64 Lemma: The projection $\pi\colon G^{\star\infty} \longrightarrow G^{\star\infty}/G$ is a numerable principal G-bundle.

7.65 Lemma: If $\pi'\colon E \to E/G$ is a numerable principal G-bundle, then there is a G-equivariant map $E \to G^{\star\infty}$.
(Note: the proof of this lemma requires the use of Proposition 7.47.)

7.66 Lemma: If E is any G-space, then all G-equivariant maps $E \to G^{\star\infty}$ are G-homotopic.

- Note that these three lemmas jointly imply Theorem 7.55.

7.67 Proposition: For any topological group G, the Milnor G-space $G^{\star\infty}$ is contractible.
(This follows very easily from the proof of Lemma 7.66.)

7.68 Proposition: For any topological group G, its classifying space BG is path-connected, and for all $n \geqslant 1$, we have $\pi_n(BG) \cong \pi_{n-1}(G)$.

7.69 Remark: In fact, by a slightly more careful argument, we can show that $\Omega BG \simeq G$.

7.70 Examples: If G is discrete, then $BG \simeq K(G,1)$. So we have:
- $B(\mathbb{Z}/2) \simeq \mathbb{RP}^\infty$,
- $B\mathbb{Z} \simeq S^1$.

Also, Proposition 7.68 tells us that the classifying space of S^1 is a $K(\mathbb{Z},2)$, so we also have:
- $BS^1 \simeq \mathbb{CP}^\infty$.

28. Wednesday 30 January

- Our next aim is to characterise universal principal G-bundles as those whose total space is contractible. First, we need yet another fact about partitions of unity.

7.71 Theorem: Let $f\colon E_1 \to E_2$ be a morphism in Top$/B$, i.e., a commutative triangle

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd} \tag{25}$$

in Top, and let $\mathcal{U}$ be a numerable open cover of B. Assume that the restriction of f to $p_1^{-1}(U) \to p_2^{-1}(U)$ is a fibrewise homotopy equivalence (i.e., homotopy equivalence in Top$/B$) for all $U \in \mathcal{U}$. Then f is a fibrewise homotopy equivalence.

7.72 Corollary: Let (25) be a morphism in Top$/B$ and now assume that p_1 and p_2 are fibre bundles that are *simultaneously numerable* (there exists a numerable open cover of B that locally trivialises both of them). Assume that the restriction of f to the fibres $p_1^{-1}(b) \to p_2^{-1}(b)$ is a homotopy equivalence for all $b \in B$. Then f is a fibrewise homotopy equivalence.

7.73 Corollary: If $p\colon E \to B$ is a numerable bundle with fibre $F \simeq *$, then p is a homotopy equivalence.

7.74 Remark: This is analogous to the fact that, if $p\colon E \to B$ is a Serre fibration with weakly contractible fibres, then it is a weak homotopy equivalence (which follows directly from the long exact sequence of homotopy groups). One might expect the analogue to state that, if $p\colon E \to B$ is a Hurewicz fibration with contractible fibres, then p is a homotopy equivalence — but this is actually false (see Counterexample 7.75). Instead, the analogue is provided by Corollary 7.73, assuming the stronger (by Corollary 7.42) hypothesis that p is a numerable bundle.

7.75 The map $\mathbb{Q}^\delta \to \mathbb{Q}$, where $\mathbb{Q}^\delta$ denotes the rational numbers with the discrete topology and the map is the identity of the underlying sets, is a Hurewicz fibration with contractible (in fact point) fibres, but not a homotopy equivalence.

7.76 Theorem: Let $p\colon E \to B$ be a numerable principal G-bundle. Then p is universal if and only if E is contractible.

- Now we will apply this theorem to construct a universal principal G-bundle in the case $G = \mathrm{Diff}(M)$, for a smooth manifold M.

7.77 Definition: A left G-space X is *G-locally continuously transitive* (G-lct) if each point $x \in X$ has an open neighbourhood U and a continuous map $\gamma\colon U \to G$ sending x to id_G such that, for each $u \in U$, we have $\gamma(u).x = u$.

7.78 Proposition:
- (a) If $f\colon X \to Y$ is G-equivariant and Y is G-lct, then f is a fibre bundle.
- (b) Let X be a left G-space and a right H-space, and assume that these actions commute. Assume that the quotient X/H is G-lct, and that $x.-\colon H \to X$ is a topological embedding for each $x \in X$. Then the quotient map $X \to X/H$ is a principal H-bundle.

7.79 Definition: Let L, M be two smooth manifolds (without boundary), where L is compact. We write $\mathrm{Emb}(L,M)$ for the space of all smooth embeddings $L \hookrightarrow M$, equipped with the Whitney C^∞ topology. We note that this topology (assuming that L is compact!) is paracompact, Hausdorff and second-countable. Moreover, there are continuous group actions

$$\mathrm{Diff}_c(M) \curvearrowright \mathrm{Emb}(L,M) \curvearrowleft \mathrm{Diff}(L),$$

where $\mathrm{Diff}_c(M)$ is the subgroup of *compactly-supported* diffeomorphisms of M. Additionally, for any embedding $e\colon L \hookrightarrow M$, the map $\mathrm{Diff}(L) \to \mathrm{Emb}(L, M)$ given by reparametrising e is a topological embedding.

7.80 Theorem:
- (a) $\mathrm{Emb}(L, M)$ is $\mathrm{Diff}_c(M)$-lct.
- (b) $\mathrm{Emb}(L, M)/\mathrm{Diff}(L)$ is $\mathrm{Diff}_c(M)$-lct.

7.81 Corollary:
- (a) Fix $e \in \mathrm{Emb}(L, M)$. Then the map $\mathrm{Diff}_c(M) \to \mathrm{Emb}(L, M)$ given by post-composition is a fibre bundle (and therefore a Hurewicz fibration). The fact that we may lift paths up this map is equivalent to the Isotopy Extension Theorem. The fact that we may lift homotopies up this map is a parametrised version of the Isotopy Extension Theorem.
- (b) The quotient map $\mathrm{Emb}(L, M) \to \mathrm{Emb}(L, M)/\mathrm{Diff}(L)$ is a principal $\mathrm{Diff}(L)$-bundle.

7.82 Facts:
- The quotient space $\mathrm{Emb}(L, M)/\mathrm{Diff}(L)$ is paracompact Hausdorff.
- Thus:
$$\pi_{L,M} : \mathrm{Emb}(L, M) \longrightarrow \mathrm{Emb}(L, M)/\mathrm{Diff}(L)$$
is a numerable principal $\mathrm{Diff}(L)$-bundle.
- These results all go through also with $M = \mathbb{R}^\infty$ instead of a finite-dimensional manifold.

7.83 Lemma: The embedding space $\mathrm{Emb}(L, \mathbb{R}^\infty)$ is contractible.
(The proof is very analogous to the proof of Proposition 7.67, that the Milnor G-space $G^{\star\infty}$ is contractible.)

7.84 Corollaries:
- The universal principal $\mathrm{Diff}(L)$-bundle is:
$$\pi_{L,\mathbb{R}^\infty} : \mathrm{Emb}(L, \mathbb{R}^\infty) \longrightarrow \mathrm{Emb}(L, \mathbb{R}^\infty)/\mathrm{Diff}(L).$$
- $B\mathrm{Diff}(L) \simeq \mathrm{Emb}(L, \mathbb{R}^\infty)/\mathrm{Diff}(L) = \{A \subset \mathbb{R}^\infty \mid A \cong L\}$, the space of smooth submanifolds of $\mathbb{R}^\infty$ that are abstractly diffeomorphic to L.
- A *smooth L-bundle* means a fibre bundle with fibre L and structure group $\mathrm{Diff}(L)$. The *universal smooth L-bundle* is therefore:
$$\mathrm{Emb}(L, \mathbb{R}^\infty) \times_{\mathrm{Diff}(L)} L \longrightarrow B\mathrm{Diff}(L),$$
whose total space may be identified as the space $\{(A, x) \in B\mathrm{Diff}(L) \times \mathbb{R}^\infty \mid x \in A\}$ of pointed submanifolds of $\mathbb{R}^\infty$ that are abstractly diffeomorphic to L, and the projection map simply forgets the basepoint. This is often called the *tautological L-bundle*, since its fibre over a point A in the base space is equal to A.

- Finally, we apply Theorem 7.76 to describe the universal vector bundle of rank n, and say a few words about topological K-theory.
- Recall that a *vector bundle* (over $\mathbb{R}$ or $\mathbb{C}$, but the notion can be made much more general) is a fibre bundle with fibre $\mathbb{R}^n$ and structure group $GL_n(\mathbb{R})$ for some $n \in \mathbb{N}$ (or the same with $\mathbb{C}$ instead of $\mathbb{R}$).

7.85 Classification:
- The universal principal $GL_n(\mathbb{R})$-bundle is
$$* \simeq \mathrm{InjLin}(\mathbb{R}^n, \mathbb{R}^\infty) = EGL_n(\mathbb{R}) \longrightarrow BGL_n(\mathbb{R}) = \mathrm{Gr}_n(\mathbb{R}^\infty),$$
whose total space is the space of injective linear maps from $\mathbb{R}^n$ into $\mathbb{R}^\infty$, or equivalently the space of linearly independent n-tuples in $\mathbb{R}^\infty$. It is not hard to see that this is contractible, using the same tricks as before. The classifying space $BGL_n(\mathbb{R})$ is then the space of n-dimensional linear subspaces of $\mathbb{R}^\infty$, i.e., the *Grassmannian* $\mathrm{Gr}_n(\mathbb{R}^\infty)$.
- The universal real vector bundle of rank n is therefore
$$\mathrm{InjLin}(\mathbb{R}^n, \mathbb{R}^\infty) \times_{GL_n(\mathbb{R})} \mathbb{R}^n \longrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty),$$
whose total space may be identified as the space $\{(V, x) \in \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty \mid x \in V\}$ of n-dimensional subspaces of $\mathbb{R}^\infty$ together with a point on this subspace, and the projection map forgets this point. Analogously to 7.84 above, this is often called the *tautological rank-n real vector bundle*, since its fibre over the point V is the vector space V itself.

- So in particular we have a bijection $\mathbb{R}\mathrm{Vect}^n_N(B) \cong [B, \mathrm{Gr}_n(\mathbb{R}^\infty)]$, where $\mathbb{R}\mathrm{Vect}^n_N(B)$ denotes the set of isomorphism classes of numerable rank-n real vector bundles over B.
- The analogous results also hold for complex vector bundles.

7.86 K-theory:

- Given two vector bundles $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ over the same base space, we may take their product and then pull back along the diagonal $\Delta\colon B \to B \times B$ to obtain the *Whitney sum*:

$$p_1 \oplus p_2 = \Delta^*(p_1 \times p_2)\colon \Delta^*(E_1 \times E_2) \longrightarrow B.$$

- This turns $\coprod_{n\in\mathbb{N}} \mathbb{R}\mathrm{Vect}^n_N(B)$ into a commutative monoid (the identity element is the trivial bundle $\mathrm{id}\colon B \to B$).
- Using the *Grothendieck group* $\mathrm{Gro}(M)$ of a monoid M, we may therefore define:

$$\begin{aligned} KO(B) &= \mathrm{Gro}\big(\textstyle\coprod_{n\in\mathbb{N}} \mathbb{R}\mathrm{Vect}^n_N(B)\big) \\ KU(B) &= \mathrm{Gro}\big(\textstyle\coprod_{n\in\mathbb{N}} \mathbb{C}\mathrm{Vect}^n_N(B)\big) \end{aligned}$$

- Theorem: If B is path-connected, compact and Hausdorff, there are natural bijections:

$$\begin{aligned} KO(B) &\cong [B, \mathbb{Z} \times BO] \\ KU(B) &\cong [B, \mathbb{Z} \times BU], \end{aligned}$$

  where BO is the colimit of the classifying spaces of the orthogonal groups $O(n)$ and BU is the colimit of the classifying spaces of the unitary groups $U(n)$.

7.87 Bott periodicity:

- The functor KO is the degree-zero part of a cohomology theory (and similarly for KU). This follows from the following steps:
- One can first check that KO satisfies the exactness and the wedge axioms.
- One may then tautologically extend this to the part of a cohomology theory defined in negative degrees: For any cohomology theory h, the functor h^n is determined by h^{n+1} via the chosen isomorphism $h^n \cong h^{n+1} \circ \Sigma$, where Σ is the suspension endofunctor. We may therefore take the functor in degree $n < 0$ to be $KO \circ \Sigma^{-n}$ and the suspension isomorphism to be the identity. However, it is not clear how to extend this into positive degrees. The first step would be to try to find a functor h^1 (satisfying the exactness and wedge axioms) such that $KO \cong h^1 \circ \Sigma$.
- One equivalent statement of the periodicity theorem of Bott is that the sequence of functors $KO \circ \Sigma^{-n}$ for $n < 0$ is *periodic* (in fact precisely 8-periodic, and the analogous sequence for KU is exactly 2-periodic).
- Hence we may extend this sequence to $n > 0$ just using periodicity, and we automatically obtain a cohomology theory of which KO is the functor in degree zero (and similarly for KU).

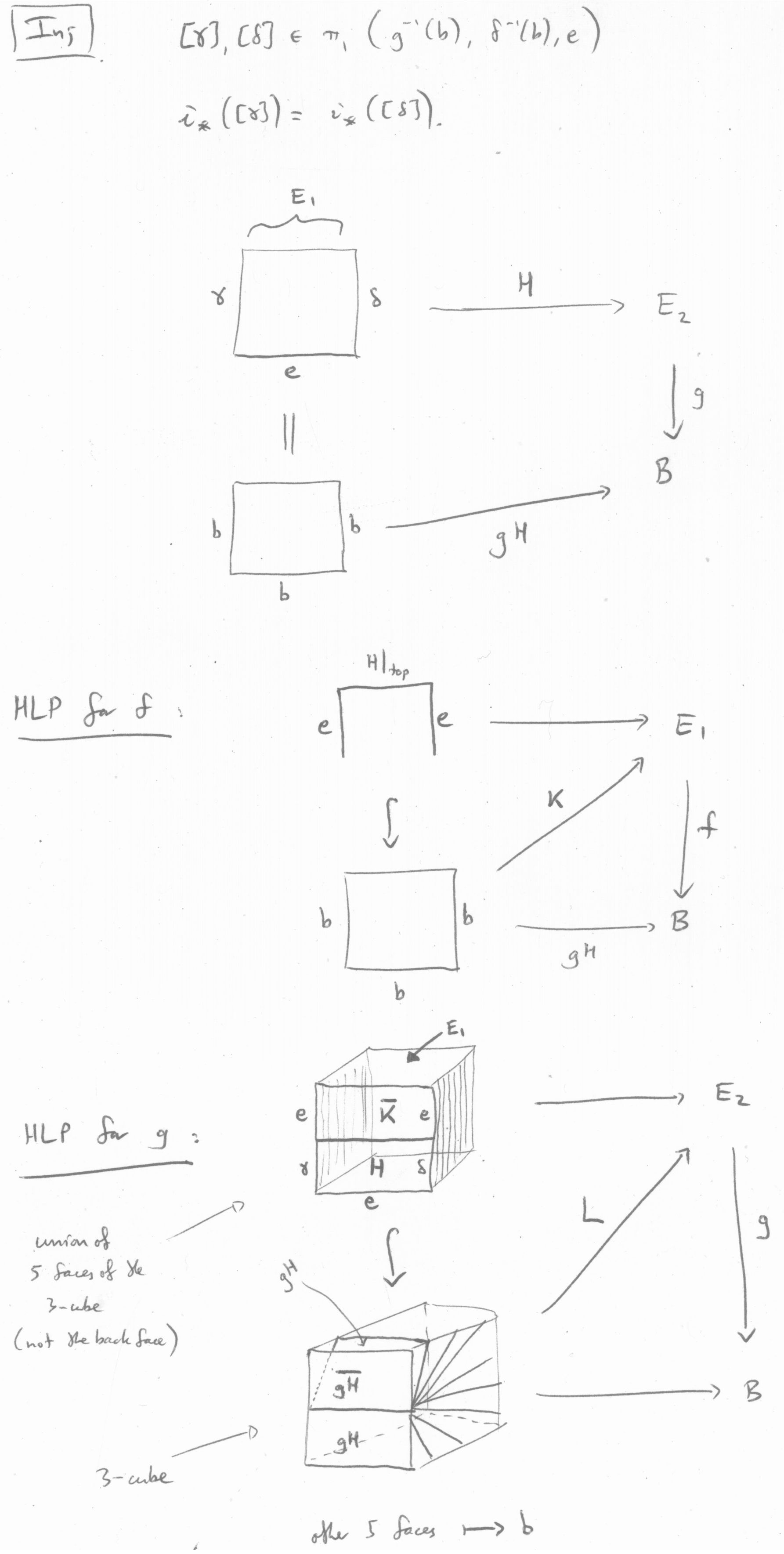

Figure 1 Proof of one direction of Lemma 3.2 (page 19) for $n = 1$.

∞. Some relevant references

- **1. Review**
  - A. Hatcher (*Algebraic topology*), chapter 4.1
  - A. Hatcher (*Algebraic topology*), pages 366 onwards (for the Hurewicz theorem)
  - M. Arkowitz (*Introduction to homotopy theory*), chapter 2 (H-spaces and co-H-spaces)
- **2. Fibrations and cofibrations**
  - T. tom Dieck (*Algebraic topology*), chapter 5
  - J. P. May (*A concise course in Algebraic Topology*), chapters 6–8
  - S. Mitchell (*Notes on Serre fibrations*)
  - N. Strickland (*The category of CGWH spaces*)
- **3. The Blakers-Massey theorem**
  - T. tom Dieck (*Algebraic topology*), chapter 6
- **4. Representability theorems**
  (including (co)homology theories, spectra, Moore-Postnikov towers)
  - A. Dold (*Halbexakte Homotopiefunktoren*), chapter 16
  - R. Switzer (*Algebraic topology — homotopy and homology*), chapters 7–9
  - A. Hatcher (*Algebraic topology*), chapter 4.3
  - E. H. Brown (*Cohomology theories*), Annals of Mathematics (1962)
  - J. F. Adams (*A variant of E. F. Brown's representability theorem*), Topology (1971)
- **5. Quasifibrations and the Dold-Thom theorem**
  - A. Dold, R. Thom (*Quasifaserungen und Unendliche Symmetrische Produkte*), Annals of Mathematics (1958)
  - A. Hatcher (*Algebraic topology*), appendix 4.K
- **6. Serre classes and rational homotopy groups of spheres**
  - T. tom Dieck (*Algebraic topology*), chapter 20
  - T. tom Dieck (*Algebraic topology*), chapter 17.7–17.9 (for the Wang sequence, Thom isomorphism theorem and Gysin sequence)
- **7. Principal bundles, vector bundles, classifying spaces**
  - N. Steenrod (*The topology of fibre bundles*), sections 2–3
  - T. tom Dieck (*Algebraic topology*), chapters 13 (partitions of unity) and 14 (bundles)
  - J. Milnor (*Construction of universal bundles, II*), Annals of Mathematics (1956)
  - For spaces of smooth embeddings:
    - R. Palais (*Local triviality of the restriction map for embeddings*), Commentarii Mathematici Helvetici (1960)
    - J. Cerf (*Topologie de certains espaces de plongements*), §II.2.2, Bulletin de la Société Mathématique de France (1961)
    - E. Lima (*On the local triviality of the restriction map for embeddings*), Commentarii Mathematici Helvetici (1963–1964)

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