**Revision sheet**

The following exercises cover some of the topics of the lecture and will be discussed in the exercise sessions on 1 February.

**Exercise 1.** Decide whether the following statements are true or false.

(1) The suspension homomorphism $\Sigma\colon \pi_3(S^2) \to \pi_4(S^3)$ is injective.

(2) If $p\colon E \to B$ is a Serre fibration over a path-connected space $B$, then all fibres $p^{-1}(b)$ for $b \in B$ are weakly homotopy equivalent.

(3) For all CGWH spaces $X, Y$ and $Z$, there is a natural homeomorphism

$$\mathrm{Map}(X \times Y, Z) \longrightarrow \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

(4) Every basepoint-preserving map $\mathbb{R}P^\infty \to \mathbb{C}P^\infty$ is based homotopic to a constant map.

(5) For any based CW-complex $Z$, the functor $\langle -, Z\rangle$ from the category of based CW-complexes to the category of pointed sets is half-exact.

(6) Let $h_*$ and $k_*$ be reduced homology theories defined on the category of based CW-complexes such that $h_i(S^0) \cong k_i(S^0)$ for all $i \in \mathbb{Z}$. Then any isomorphism of groups $h_0(S^0) \to k_0(S^0)$ extends to an isomorphism of homology theories $h_* \cong k_*$.

(7) Let $X$ and $Y$ be CW-complexes. Then there exists a weak equivalence $SP(X \vee Y) \to SP(X) \times SP(Y)$.

(8) Let $p$ be a prime number. Then the class of all finitely generated abelian groups $A$ such that $a \mapsto p \cdot a\colon A \to A$ is invertible forms a Serre class.

(9) Let $G$ be a discrete group and $E$ a free right $G$-space. Then the orbit map $E \to E/G$ is a principal $G$-bundle.

(10) There are infinitely many pairwise non-isomorphic principal $\mathbb{Z}$-bundles over the base space $S^1$.

**Exercise 2.** Assume that the following diagram is a pushout diagram in the category of topological spaces.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r, "g"'] & D \end{tikzcd}$$

Prove that, if $i$ is a cofibration, then $j$ is a cofibration. Is this statement true if we work instead in the category CGWH?

**Exercise 3.** Let $X$ be a path-connected space and let $\Sigma X$ denote its reduced suspension. Show that, for $n \geqslant 3$, there are isomorphisms

$$\pi_n(\Sigma X, X) \cong \pi_n(\Sigma X) \times \pi_{n-1}(X).$$

**Exercise 4.** Show that for any two abelian groups $A, B$ and any $n \geqslant 2$ there is a bijection

$$\mathrm{Hom}(A, B) \cong \langle K(A, n), K(B, n) \rangle$$

between the set of group homomorphisms $A \to B$ and the set of based homotopy classes of based maps $K(A, n) \to K(B, n)$ between the corresponding Eilenberg-MacLane spaces.

**Exercise 5.** Compute $\pi_*(\mathbb{C}P^n) \otimes \mathbb{Q}$.

**Exercise 6.** Find maps $f_i \colon X_i \to Y_i$ for $i = 1, 2$ between simply-connected CW-complexes such that:

(a) $f_1$ induces a surjection on $\pi_*$ but not on $H_*$,

(b) $f_2$ induces a surjection on $H_*$ but not on $\pi_*$.

(*Hint:* Only consider spheres and projective spaces. For (b), use Exercise 5 to deduce that $\pi_7(\mathbb{C}P^2)$ is finite.)