S4D2 – Graduate Seminar on Topology
Braid groups and configuration spaces

S2D5 – Hauptseminar Niedrigdimensionale Topologie
Zopfgruppen und Konfigurationsräume

Martin Palmer-Anghel // Summer semester 2019 // Wed 12pm (c.t.), seminar room N0.007

Summary. The aim of this seminar is to introduce the *braid groups* and to study their various different facets, including their relation to *configuration spaces*, as well as to mapping class groups of surfaces and knot theory. The overall outline will be:

- Introduction and relation to configuration spaces (talks 1–4).
- Connections with automorphism groups of free groups and mapping class groups of surfaces (talks 5–7).
- Connections to knot theory (talks 8–9).
- Representations of braid groups via mapping class groups (talks 10–12).
- Homology and cohomology of the braid groups (talks 13–14).

Website.

- math.uni-bonn.de/people/palmer/Zopf.html, or
- mdp.ac/teaching/19-braids-configurations.html

Prerequisites for the seminar.

- The lecture course *Einführung in die Geometrie und Topologie*, in particular the fundamental group and covering space theory.
- The lecture course *Topologie I*, in particular homology.

Seminar requirements. The requirements for this seminar are (a) regular participation in the seminar and (b) giving one talk (or more, if you wish). You should discuss the topic of your talk with me at least a week in advance (see the website for my office hours), by which time you should already have an overall plan for your talk. Talks should be 90 minutes long, including time for questions (so plan for 75–80 minutes).

List of talks.

1. Definitions of braid groups
2. Fibre bundles, classifying spaces and homotopy groups
3. The pure braid group and the centre of the braid group
4. Configuration spaces and spaces of polynomials
5. Automorphisms of free groups
6. Braid groups as mapping class groups
7. Surface braid groups and the Birman exact sequence
8. Braid groups and links
9. Markov's theorem
10. Homological representations of braid groups – the Burau representation
11. The Lawrence-Krammer-Bigelow representations of braid groups
12. Linearity of the braid groups
13. Mod-2 cohomology of braid groups
14. Integral homological stability for braid groups

Brief descriptions of the talks.

1. Definitions of braid groups. (03.04)
(*Speaker:* **Arne Beines**)
Introduce the braid groups via braid diagrams and Reidemeister moves. Sketch the proof of Theorem 1.6 of [KT], relating Reidemeister moves and isotopies of braid diagrams. Describe a finite presentation of the braid groups via Theorem 1.12 of [KT].
References. [KT] sections 1.1 and 1.2.

2. Fibre bundles, classifying spaces and homotopy groups. (10.04)
(*Speaker:* **Nicolas Schmitt**)
Give an overview of the theory of fibre bundles, covering spaces and classifying spaces of (discrete) groups. Also define the homotopy groups $\pi_n(-)$ and describe the long exact sequence of homotopy groups associated to a fibre bundle. The main goal of this talk is to give a clear account of this theory, without going too deeply into the details of many proofs.
References. [DK] chapter 4, [Hat] chapters 1.3 and 1.B plus pp. 340ff and 375ff

3. The pure braid group and the centre of the braid group. (17.04)
(*Speaker:* **Jonas Nehme**)
Define the *pure braid groups*, and prove several group-theoretic facts about them, assuming Theorem 1.16 of [KT], which will be proved in the next talk. Namely, prove that the pure braid groups P_n are torsion-free and residually finite, and that the abelianisation of P_n is free of rank $\binom{n}{2}$. Also calculate the *centre* of the braid group B_n and deduce that braid groups on different numbers of strands cannot be isomorphic.
References. [KT] section 1.3

4. Configuration spaces and spaces of polynomials. (24.04)
(*Speaker:* **Erik Babuschkin**)
Introduce *configuration spaces* and the *Fadell-Neuwirth fibre bundles*[1] (Lemma 1.27 of [KT]), and use these to prove that configuration spaces on aspherical surfaces are aspherical. Give another equivalent definition of (pure) braid groups in terms of configuration spaces. Then prove Theorem 1.16 of [KT], which was assumed in the previous talk, and give a topological proof of the fact that the braid groups B_n are torsion-free. Describe a model for the classifying space of B_n as a space of polynomials.
References. [KT] section 1.4

5. Automorphisms of free groups. (01.05)
(*Speaker:* **Johanna Luz**)
Prove that the braid group B_n embeds into the group $\mathrm{Aut}(F_n)$ of automorphisms of the free group on n letters (Theorem 1.31 of [KT]). Explain what is meant by the *word problem* for a group G with chosen generating set S, and give a solution to the word problem for the braid group B_n with its standard set of generators.
References. [KT] section 1.5

6. Braid groups as mapping class groups. (08.05)
(*Speaker:* **Nicolai Gerber**)
Introduce *mapping class groups* of manifolds, and describe the elements called “half-twists” in the case of an oriented surface. Give a sketch of a proof that the braid groups B_n are isomorphic to certain mapping class groups (Theorem 1.33 of [KT]).
References. [KT] sections 1.6 and 1.7

[1] Note that [KT] use the terminology “locally trivial fibration” instead of “fibre bundle”.

7. Surface braid groups and the Birman exact sequence. (15.05)

(*Speaker:* **Avital Berry**)

Define the *surface braid groups* $B_n(S)$ for a given surface S, and construct the generalised Birman exact sequence (Theorem 9.1 of [FM]), which shows that $B_n(S)$ is a subgroup of a certain mapping class group, as long as S satisfies a certain condition (in particular if the Euler characteristic of S is negative or if S is the closed 2-disc). The case $S = D^2$ generalises the result of the previous talk via the Alexander lemma (Lemma 2.1 of [FM]). First prove the special case of the Birman exact sequence given in Theorem 4.6 of [FM] and then explain how to extend this to the general case. If there is time, show also that the braid groups $B_n(S^2)$ on the 2-sphere are extensions of mapping class groups by $\mathbb{Z}/2$.

References. [FM] sections 4.2 and 9.1

8. Braid groups and links. (22.05)

(*Speaker:* **Roman Pleskovsky**)

Introduce links, link diagrams and their Reidemeister moves. Prove that two n-strand braids have isotopic closures in the solid torus if and only if they are conjugate in B_n (Theorem 2.1 of [KT]). Then prove that every link in $\mathbb{R}^3$ is isotopic to the closure of some braid (Alexander's theorem; Theorem 2.3 of [KT]).

References. [KT] sections 2.1–2.3

9. Markov's theorem. (29.05)

(*Speaker:* **Koen van Greevenbroek**)

Prove that two braids have isotopic closures in $\mathbb{R}^3$ if and only if they are related by a finite sequence of so-called *Markov moves*. This is Markov's theorem; Theorem 2.8 of [KT]. The proof of this is relatively long, so you should give an account of the main ideas and select a few (but not all!) of the details to explain carefully.

References. [KT] sections 2.5–2.7

10. Homological representations of braid groups – the Burau representation. (05.06)

(*Speaker:* **Sid Maibach**)

Define the *Burau representation* of B_n, first by assigning explicit matrices to its generators (as in section 3.1.1 of [KT]) and then geometrically (as in sections 3.2.1 and 3.2.2 of [KT]), using the interpretation of B_n as a mapping class group, and prove that these two definitions are equivalent (section 3.2.5 of [KT]). Sketch the proof that these representations are not faithful for $n \geqslant 6$, via introducing the theory of *Dehn twists* on surfaces.

References. [KT] sections 3.1 and 3.2

11. The Lawrence-Krammer-Bigelow representations of braid groups. (19.06)

(*Speaker:* **Branko Juran**)

Define the *Lawrence-Krammer-Bigelow* (LKB) representation of B_n as in sections 3.5.1–3.5.3 of [KT]. This is a homomorphism $B_n \to \mathrm{Aut}_R(\mathcal{H})$, where $R = \mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ and $\mathcal{H}$ is an R-module. Sketch a proof that $\mathcal{H}$ is a free R-module of rank $\binom{n}{2}$, following section 4 of [Big]. Deduce that the braid groups B_n are *linear* (i.e. embed into $GL_r(\mathbb{F})$ for some finite r and field $\mathbb{F}$), assuming that the LKB representation $B_n \to \mathrm{Aut}_R(\mathcal{H})$ is faithful (i.e. injective).

References. [KT] sections 3.5.1–3.5.4, [Big]

12. Linearity of the braid groups. (26.06)

(*Speaker:* **Sigurður Jens Albertsson**)

Complete the proof that the braid groups are linear by proving that the LKB representation $B_n \to \mathrm{Aut}_R(\mathcal{H})$ is faithful (injective), following sections 3.5.5 and 3.6–3.7 of [KT]. This is a long and quite intricate proof, so your aim should be to give an outline of the ideas of the proof, and select just a few points to prove in more detail.

References. [KT] sections 3.5–3.7

13. Mod-2 cohomology of braid groups. (03.07)
(*Speaker:* **Ödül Tetik**)
Compute the mod-2 cohomology of the configuration space $C_n(\mathbb{R}^2)$ — which is the mod-2 cohomology of B_n, since we know that $C_n(\mathbb{R}^2)$ is a classifying space for B_n — following [Fuks], by studying a certain cell decomposition of $C_n(\mathbb{R}^2)$ and using cellular cohomology. Compute the additive structure of the cohomology, i.e. as a graded $\mathbb{F}_2$-vector space, and describe (without proof) its multiplicative structure.
References. [Fuks], [Arn]

14. Integral homological stability for braid groups. (10.07)
(*Speaker:* **Peng Hui How**)
Let M be a connected manifold with non-empty boundary (the important example for this seminar is $M = D^2$). Define a *stabilisation map* $C_n(M) \to C_{n+1}(M)$ and prove that this map induces isomorphisms on integral homology in degrees $\leqslant \frac{n}{2}$ (Proposition A.1 of [Seg]), following the proof in the appendix to §5 (pages 68–72) of [Seg].
References. [Seg] pages 68–72, [Arn]

References.

[Arn] V. I. Arnol'd, *On some topological invariants of algebraic functions*, Trans. Moscow Math. Soc., vol. 21, pp. 30–52 (1970).

[Big] S. Bigelow, *The Lawrence-Krammer representation*, from: "Topology and Geometry of Manifolds (Athens, GA, 2001)", Proc. Sympos. Pure Math., vol. 71, pp. 51–68, Amer. Math. Soc., Providence, RI (2003).

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[FM] B. Farb, D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press (2011).

[Fuks] D. B. Fuks, *Cohomologies of the group COS mod* 2, Funct. Anal. Appl., vol. 4, pp. 143–151 (1970).

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[KT] C. Kassel, V. Turaev, *Braid groups*, Graduate Texts in Mathematics, vol. 247, Springer, New York (2008).

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