Configuration spaces and homological stability

Martin Palmer // *5th July 2012*

Configuration spaces - Definition

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The (un)ordered configuration space associated to a background space M is

$$\begin{aligned} \text{(ordered)} \quad \widetilde{C}_n(M) &:= \{(p_1, \ldots, p_n) \in M^n \mid p_i \neq p_j \text{ for } i \neq j\} \\ \text{(unordered)} \quad C_n(M) &:= \{(p_1, \ldots, p_n) \in M^n \mid p_i \neq p_j \text{ for } i \neq j\}/\Sigma_n \end{aligned}$$

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- M is usually a manifold
- Think of this as the space of all configurations of n particles living inside M
- Note that the topology is such that particles cannot collide

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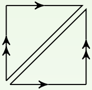
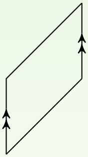
$\widetilde{C}_2(S^1) =$  $=$ 

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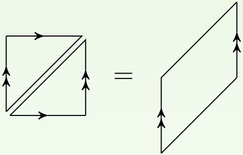
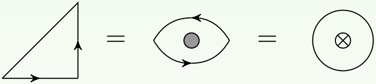
$\widetilde{C}_2(S^1) =$ 

$C_2(S^1) =$ 

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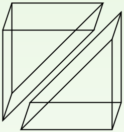
$\widetilde{C}_3(S^1) =$  $/$*glue top to bottom*

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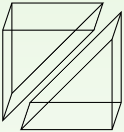
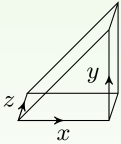
$\widetilde{C}_3(S^1) =$  $/$*glue top to bottom*

$C_3(S^1) =$ 

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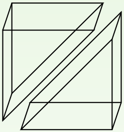

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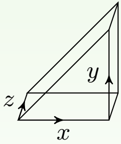

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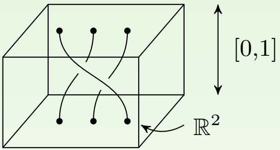

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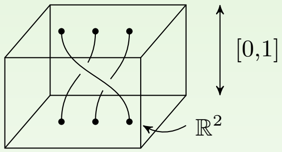

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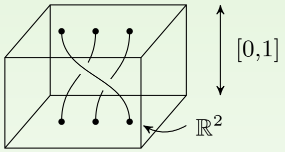

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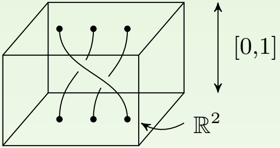

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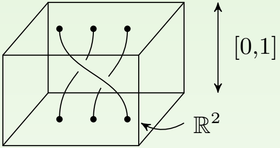

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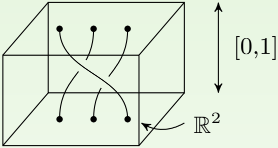

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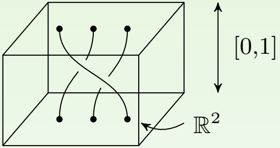

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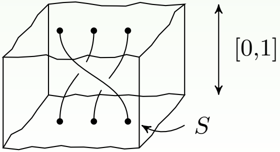

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- background space M
- parameter space X

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The (un)ordered configuration space with labels in X is

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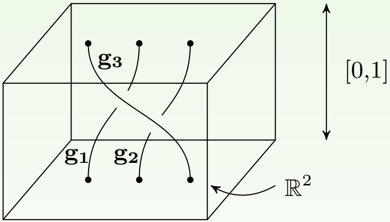
- $C_n(\mathbb{R}^2, BG) \simeq B(\beta_n \wr G)$



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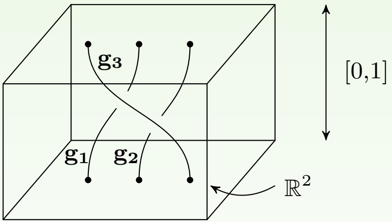

- in particular, taking $G = \mathbb{Z}$ (so $BG = S^1$) gives the ribbon braid group

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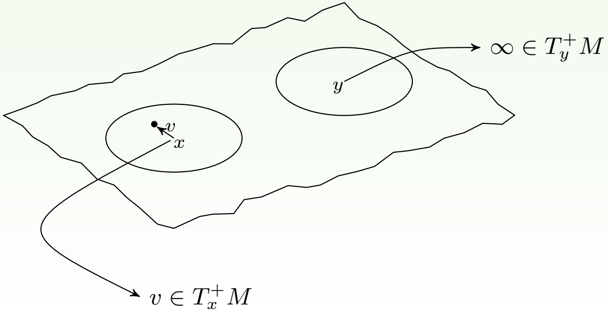

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The scanning map commutes with maps

$$C_n(M) \longrightarrow C_{n+1}(M)$$
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making $\{C_n(M)\}$ and $\{\Gamma_n(T^+M)\}$ into direct systems.

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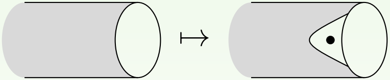

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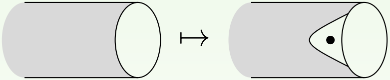

So we have a commutative ladder

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n(M) & \longrightarrow & C_{n+1}(M) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \Gamma_n(T^+M) & \longrightarrow & \Gamma_{n+1}(T^+M) & \longrightarrow & \cdots \end{array}$$

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- The limiting homology is finite-generated in each degree

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Corollary

$H_*C_n(M) \longrightarrow H_*C_{n+1}(M)$ *is an isomorphism for* $n \gg *$

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$X_n \xrightarrow{s} X_{n+1} \xrightarrow{s} \cdots$ has homological stability if for each q,

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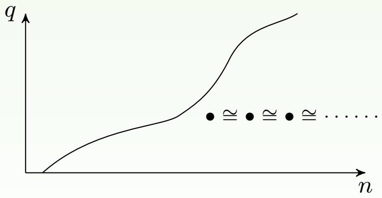
Stable range:



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- $\mathrm{Aut}(F_n)$ [Hatcher, Vogtmann, Wahl]
  limiting space $\cong_{H_*} \Omega_0^\infty S^\infty$ [Galatius]

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Theorem (Randal-Williams)

If M is a connected manifold of dimension at least 2 and is the interior of some manifold-with-boundary, and if X is path-connected, then

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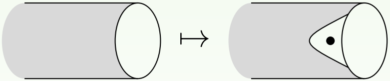
is an isomorphism for $* \leq \frac{n}{2}$.

Here the map s is



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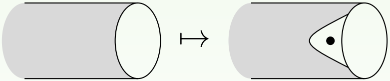

Corollaries

Homological stability for $\{\beta_n^S \wr G\}$ and $\{\Sigma_n \wr G\}$.

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- explicit presentation for $P\beta_n$ $\rightsquigarrow$ its abelianisation is $\mathbb{Z}^{\binom{n}{2}}$
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Sub-aside on Representation stability:

- Look at the rational homology $H_q(\widetilde{C}_n(M); \mathbb{Q})$ for fixed q
- We know this *doesn't* stabilise as a sequence of rational vector spaces
- But it *does* stabilise as a sequence of Σ_n-representations [Church]
- . . . meaning that their decomposition into irreducibles has a "stable description" as $n \to \infty$

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Oriented configuration space associated to M and X:

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Theorem (P)

If M is a connected manifold of dimension at least 2 and is the interior of some manifold-with-boundary, and if X is path-connected, then

$$H_*C_n^+(M,X) \xrightarrow{\;s_*\;} H_*C_{n+1}^+(M,X)$$

is an isomorphism for $* \leq \frac{n-5}{3}$.

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- The calculations also show that

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is not split-injective in general.

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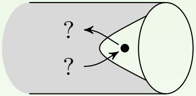

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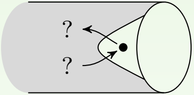

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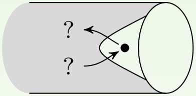

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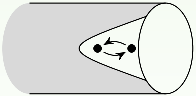

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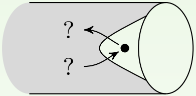

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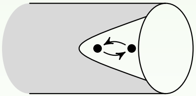

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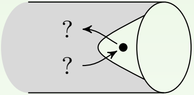

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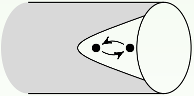

- The inductive argument now works, but using an older inductive hypothesis at each step $\rightsquigarrow$ smaller rate of stability.

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You need an appropriate notion of *finite-degree coefficient system* in each case.

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The Σ_n result generalises to:

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For any coefficient system of $\pi_1 C_n(M, X)$-modules V_n of degree d, and M and X as before,

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- Note: such coefficient systems do *not* include the sequence of coefficients V of the previous slide.
- The theorem allows systems of $\pi_1 C_n(M, X)$-modules which *don't* necessarily come from a system of Σ_n-modules via the projection $\pi_1 C_n(M, X) \twoheadrightarrow \Sigma_n$.