Homology of *symmetric diffeomorphism groups* of manifolds and *diffeomorphism groups of manifolds with singularities*

Martin Palmer — Université Paris XIII

Topology of Manifolds, Lisbon — 29 June 2016

Homological stability for families of spaces

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- Configuration spaces, moduli spaces of manifolds, ...

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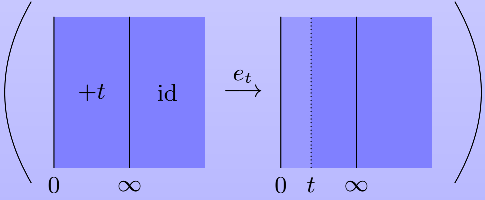

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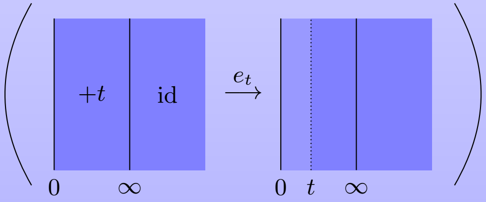

$\rightarrowtail$ one-parameter family of embeddings $[0, \infty) \longrightarrow \mathrm{Emb}(M, M)$

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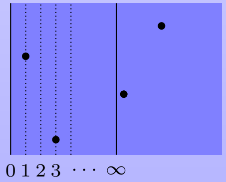

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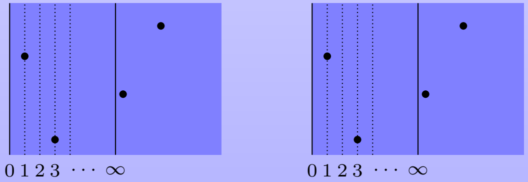

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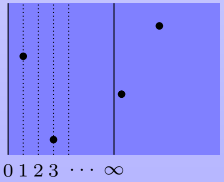

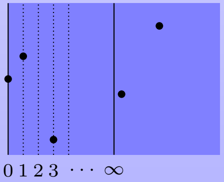

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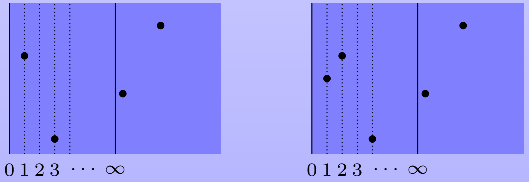

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$$\mathrm{conf}_{\Sigma_g}(M) = \mathrm{Emb}(\Sigma_g, M)/\mathrm{Diff}^+(\Sigma_g),$$

when $\pi_1(M) = 0$ and $\dim(M) \geqslant 5$.

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If $\dim(P) \leqslant \frac{1}{2}(\dim(M) - 3)$ *and* $G \leqslant \mathrm{Diff}(P)$ *is open or trivial then*

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Symmetric diffeomorphism groups

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- Generalises a theorem of Tillmann

$\longleftrightarrow$ $P =$ point and the 'usual' $\sharp$

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Then:
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Thank you for your attention