Homological stability for $\left\{ \begin{array}{l} \text{moduli spaces of disconnected submanifolds} \\ \& \\ \text{symmetric diffeomorphism groups} \end{array} \right.$

Plan

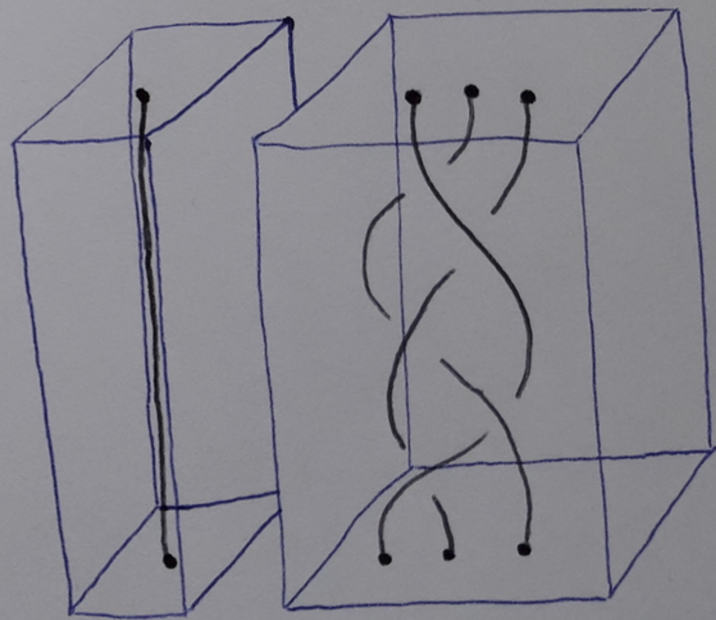
- Braid groups
- Configuration spaces of points
- Moduli spaces of disconnected submanifolds
- Symmetric diffeomorphism groups
- Diffeomorphism groups of manifolds with conical singularities

Braid groups

$$C_n(\mathbb{R}^2) = \{(x_1,\dots,x_n) \in (\mathbb{R}^2)^n \mid x_i \neq x_j \text{ for } i \neq j\} / \Sigma_n$$

<u>aspherical</u>

$$H_* C_n(\mathbb{R}^2) \cong H_* \underbrace{\pi_1 C_n(\mathbb{R}^2)}_{B_n}$$



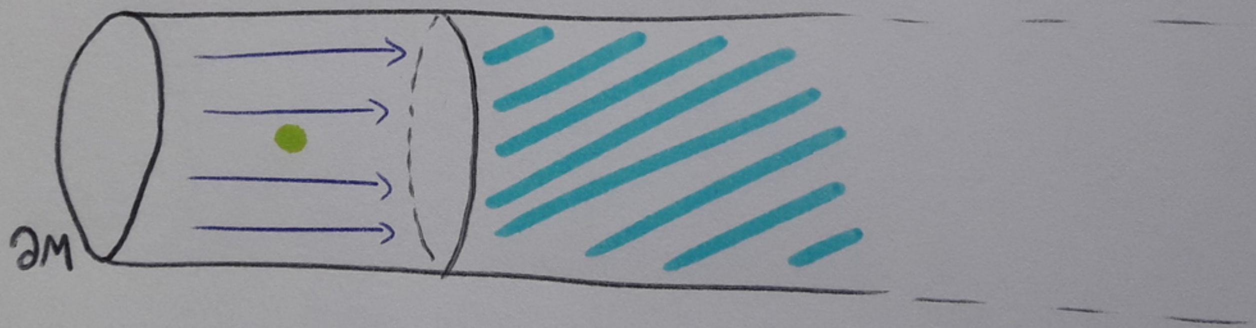
Arnol'd

$$\boxed{H_* B_n \cong H_* B_{n+1} \quad \text{for } n \geqslant 2*}$$

Configuration spaces

$$C_n(M) = \{(x_1, \dots, x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\} / \Sigma_n$$

M : connected manifold with $\partial M \neq \phi$



$$C_n(M) \longrightarrow C_{n+1}(M)$$

McDuff, Segal

$$\boxed{H_* C_n(M) \xrightarrow{\cong} H_* C_{n+1}(M) \quad \text{for } n \geqslant 2*}$$

Q: What about $\pi_1 C_n(M) \longrightarrow \pi_1 C_{n+1}(M)$?

- $\dim(M) = 2$: $C_n(M)$ aspherical
- $\dim(M) \geq 3$: $C_n(M)$ not aspherical, but

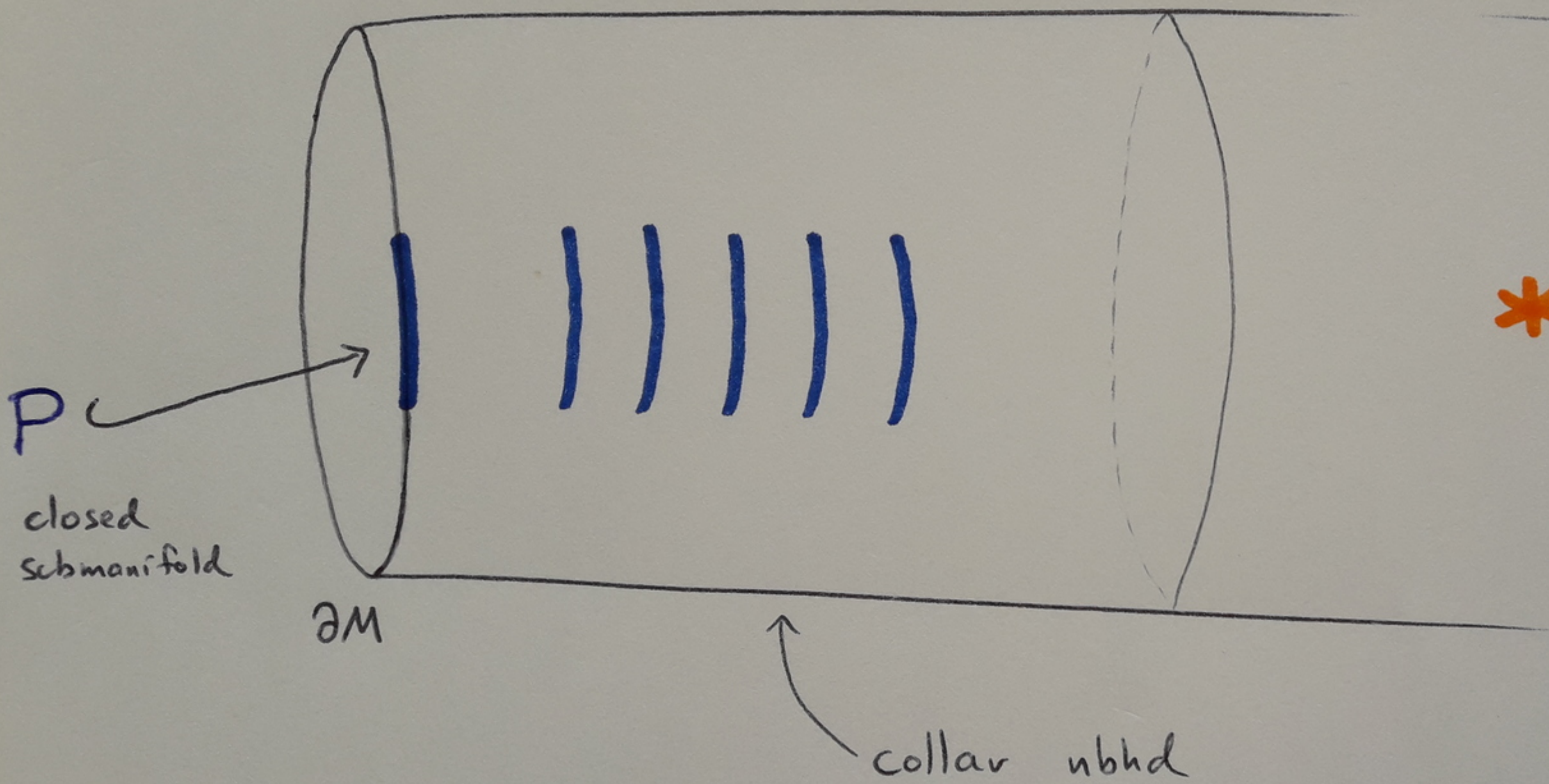
$$\pi_1 C_n(M) \cong \pi_1(M) \wr \Sigma_n \qquad \left(= \pi_1(M)^n \rtimes \Sigma_n \right)$$

$$\rightsquigarrow$$

$$B(\pi_1(M) \wr \Sigma_n) = C_n(\mathbb{R}^\infty; B\pi_1(M))$$

McDuff Stable homology of $C_n(M)$

Moduli spaces of disconnected submanifolds



$$* \in \frac{\mathrm{Emb}(nP, M)}{\mathrm{Diff}(P) \wr \Sigma_n}$$

$$C_{nP}(M) \longrightarrow C_{(n+1)P}(M)$$

$C_{nP}(M)$ = path-component of $*$ in $\dfrac{\mathrm{Emb}(nP,M)}{\mathrm{Diff}(P)\wr\Sigma_n}$

Eg $\quad S^1 \hookrightarrow \partial D^3$

$$C_{nS^1}(D^3) \longrightarrow C_{(n+1)S^1}(D^3)$$

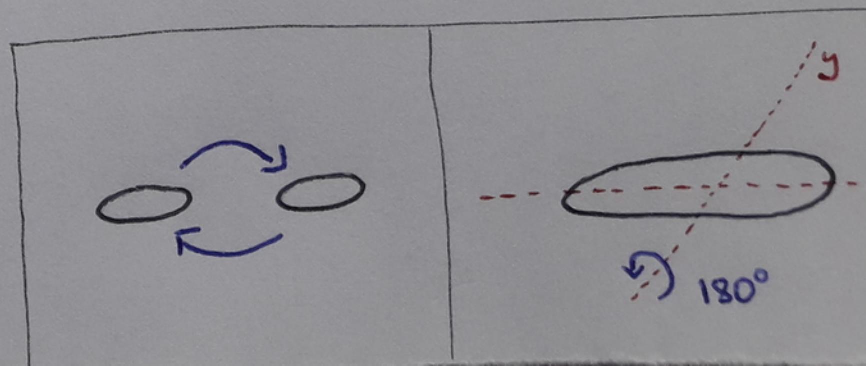
space of unlinks with n components

$\sqcup$ with unlinked copy of the unknot

$\pi_1 C_{nS^1}(D^3)$ loop braid group

Note $\quad C_{nS^1}(D^3)$ is not aspherical:

- $\simeq$ $\underbrace{\text{finite}}_{6n}$-dimensional manifold [Brendle - Hatcher]
- $\pi_1(\quad) \supseteq$ torsion

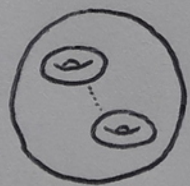


- $H_* C_{nS^1}(M)$ stabilises for $n \geqslant 2*$ [Kupers]

3-manifold

stable homology : [Griffin - Hatcher]

- $H_* \underbrace{\pi_1 C_{nS^1}(D^3)}$

$\cong$ MCG() / twists stabilises for $n \geqslant 2*$

[Hatcher - Wahl]

stable homology ?

- **Thm** (P.) $\boxed{H_* C_{nP}(M) \text{ stabilises for } n \geq 2* \quad \underline{\text{if}} \quad \dim(P) \leq \frac{1}{2}(\dim(M) - 3)}$

- + extensions to

  - $C_{nP}(M; G) \subseteq \dfrac{\mathrm{Emb}(nP, M)}{G \wr \Sigma_n}$ for G <u>open</u> $\leq \mathrm{Diff}(P)$

  - labels in a $\underset{G\text{-equivariant}}{\wedge}$ bundle $\begin{array}{c} Z \\ \downarrow \\ \mathrm{Emb}(P, M) \end{array}$

  - twisted coefficients

<u>Q</u>: stable homology?

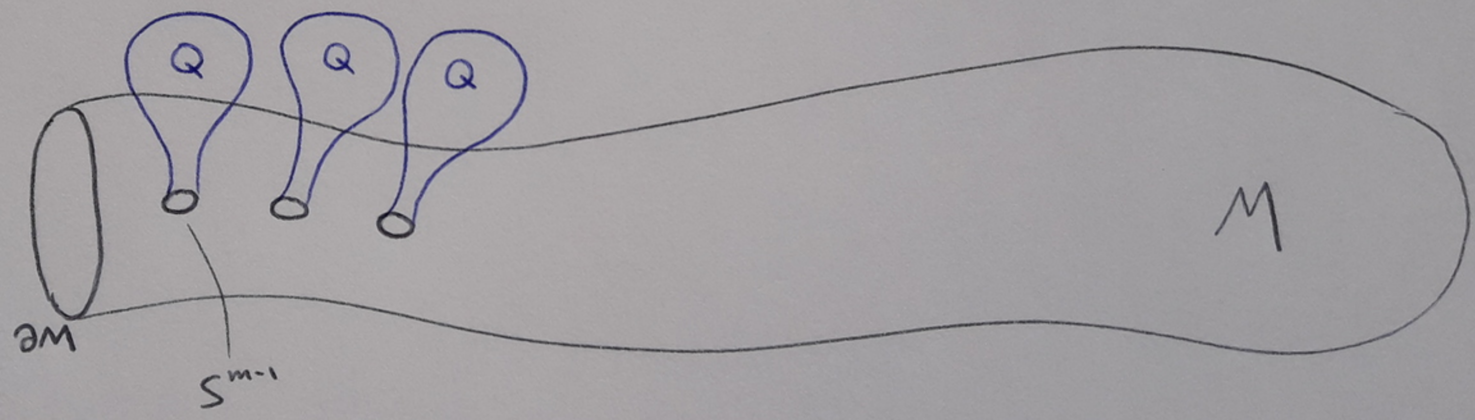
Symmetric diffeomorphism groups

M^m : connected manifold with $\partial M \neq \phi$

Q^m : based, connected manifold

(choose orientations if both are orientable)

$M \hat{\#} Q$ =



choose G closed $\leqslant O(m)$

(if · Q orientable
& · $\nexists$ o-reversing diffeo of Q then $G \leqslant SO(m)$)

Def $\Sigma_G \mathrm{Diff}(M \overset{n}{\#} Q) \leqslant \mathrm{Diff}(M \overset{n}{\#} Q)$

$$\begin{aligned} &\Sigma_G \mathrm{Diff}(M \overset{n}{\#} Q) \\ &\quad \| \\ &\{ \varphi \mid \bullet\ \varphi \text{ fixes } \partial M \text{ pointwise} \\ &\qquad \bullet\ \varphi \text{ fixes } \textstyle\coprod_n (Q \setminus \text{disc}) \text{ setwise} \\ &\qquad \bullet\ \varphi \text{ acts on } \textstyle\coprod_n S^{m-1} \text{ through } G \wr \Sigma_n \} \end{aligned}$$

Note pullback of

$$\begin{tikzcd} & \mathrm{Diff}_G(Q) \wr \Sigma_n \arrow[d] \\ \mathrm{Diff}_{G \wr \Sigma_n}(M \setminus (n \text{ discs})) \arrow[r] & G \wr \Sigma_n \end{tikzcd}$$

Tillmann $\boxed{H_* \Sigma_G \mathrm{Diff}(M \overset{n}{\#} Q) \text{ stabilises for } n \geqslant 2*}$

Parametrised connected sum

$$P \overset{e}{\hookrightarrow} M \qquad P \overset{f}{\hookrightarrow} Q$$

$$\begin{tikzcd} \nu_e \arrow[r, phantom, "\overset{\theta}{\cong}"] \arrow[dr] & \nu_f \arrow[d] \\ & P \end{tikzcd} \quad \xrightarrow[\text{bundle}]{\text{unit disc}} \quad \begin{tikzcd} D\nu_e \arrow[r, phantom, "\cong"] \arrow[dr] & D\nu_f \arrow[d] \\ & P \end{tikzcd}$$

$$\xrightarrow[\text{neighbourhoods}]{\text{tubular}} \quad D\nu_e \hookrightarrow M, \quad D\nu_f \hookrightarrow Q$$

Def

$$\boxed{M \underset{P}{\#} Q = (M \setminus D\nu_e) \underset{\theta}{\cup} (Q \setminus D\nu_f)}$$

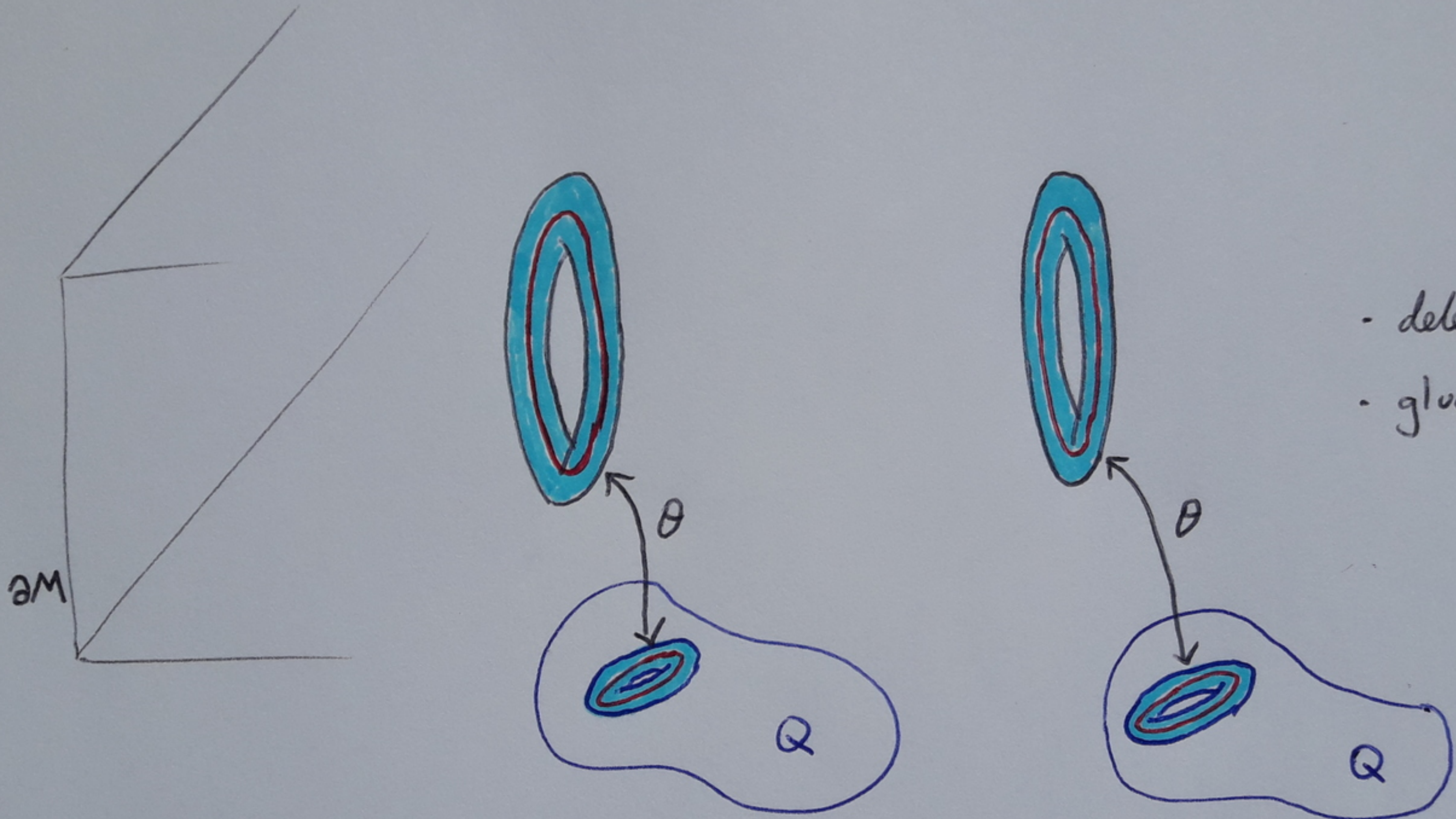
Eg

- $S^k \hookrightarrow M$ with trivialisation of its normal bundle; $S^k \hookrightarrow S^m$ $\Big\}$ $M \underset{S^k}{\#} S^m$ = k-surgery on M
- $S^1 \hookrightarrow M^3$ knot + framing; $S^1 \hookrightarrow L(p,q)$ $\Big\}$ $M \underset{S^1}{\#} L(p,q)$ = Dehn surgery on M of slope p/q

Setup

choose $P \overset{f}{\hookrightarrow} Q^m$ $\qquad D\nu_f =: T \hookrightarrow Q$

$M \underset{P}{\overset{n}{\#}} Q =$



- delete
- glue

choose $G \leqslant \mathrm{Diff}_{O(m-p)}(T \to P)$

$$\begin{array}{c} \mathrm{Diff}_{O(m-p)}(T \to P) \\ \downarrow \text{core} \\ \mathrm{Diff}(P) \end{array}$$

- $\mathrm{Diff}_G(Q) \twoheadrightarrow G$
- $\mathrm{core}(G)$ <u>open</u> $\leqslant \mathrm{Diff}(P)$
- $\underbrace{\ker(\mathrm{core}) \cap G \text{ \underline{closed}}}_{K} \leqslant \ker(\mathrm{core})$

Def $\Sigma_G \mathrm{Diff}(M \#_P^n Q) \leqslant \mathrm{Diff}(M \#_P^n Q)$

$$\| $$

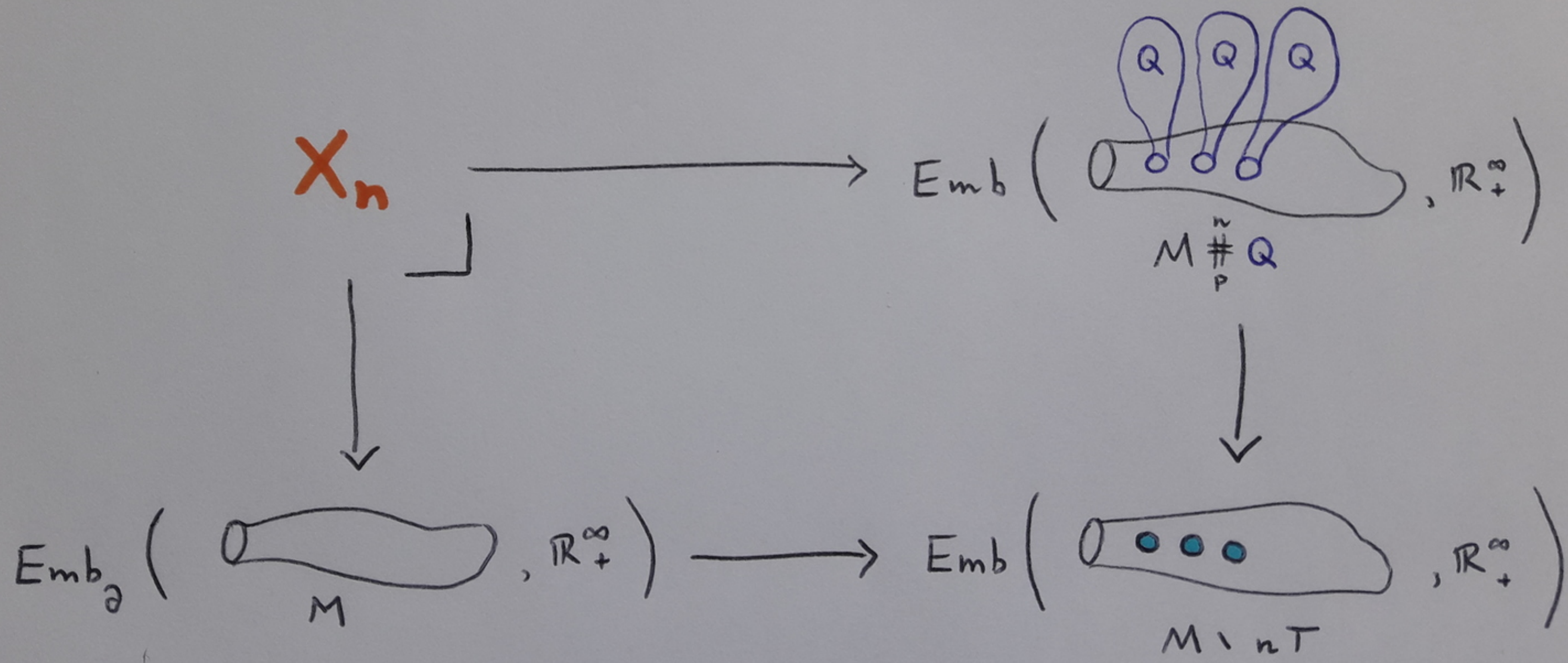
$$\left\{ \varphi \;\middle|\; \begin{array}{l} \bullet\ \varphi \text{ fixes } \partial M \text{ pointwise} \\ \bullet\ \varphi \text{ fixes } \coprod_n (Q \setminus T) \text{ setwise} \\ \bullet\ \varphi \text{ acts on } \coprod_n \partial T \text{ through } G \wr \Sigma_n \end{array} \right\}$$

- **Thm** (P.)

$\boxed{\begin{array}{l} \text{if } \dim(P) \leqslant \frac{1}{2}(\dim(M) - 3) \\ \text{then } \Sigma_G \mathrm{Diff}(M \#_P^n Q) \text{ stabilises for } n \geqslant 2* \end{array}}$

Idea of proof

- Good model for $B\Sigma_G \mathrm{Diff}(M \#_p^n Q)$:



X_n is a principal $\Sigma_G \mathrm{Diff}(M \#_p^n Q)$ - space

— Fibre bundle:

$$
\begin{tikzcd}
X_n / \Sigma_G \, \mathrm{Diff}\left(M \overset{n}{\underset{P}{\#}} Q\right) \arrow[d, "\Psi"] & \Psi^{-1}(\bullet) \arrow[l] \simeq C_{nP}(M, Z; G) \\
\mathrm{Emb}_\partial(M, \mathbb{R}^\infty_+) / \mathrm{Diff}_\partial(M) &
\end{tikzcd}
$$

$\bullet \in \mathrm{Emb}_\partial(M, \mathbb{R}^\infty_+) / \mathrm{Diff}_\partial(M)$

$$
\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, bend right] & \mathrm{Emb}(Q \setminus T, \mathbb{R}^\infty_+) / \mathrm{Diff}_K(Q) \arrow[d] \\
\mathrm{Emb}(T, M)/K \arrow[r] \arrow[d] & \mathrm{Emb}(\partial T, \mathbb{R}^\infty_+)/K \\
\mathrm{Emb}(P, M) &
\end{tikzcd}
$$

— stabilisation $\rightsquigarrow$ map of bundles

— (Serre) spectral sequence comparison

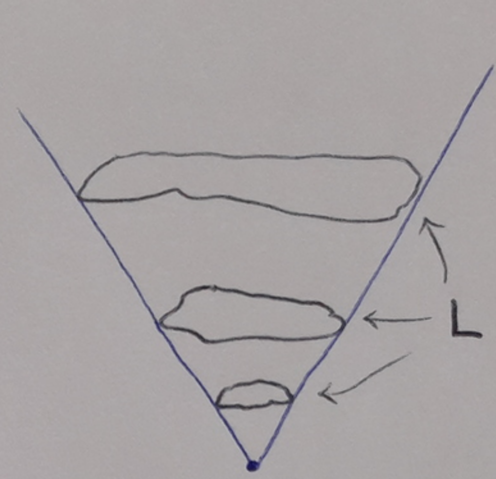
recall:

$$K = G \cap \ker\left(\mathrm{Diff}_{O(m-p)}(T \to P) \xrightarrow{\text{core}} \mathrm{Diff}(P)\right)$$

Manifolds with conical singularities

- Fix L^{m-1} closed manifold

  $G \leqslant \mathrm{Diff}(L)$

$\mathrm{cone}(L) =$ 

$$= \frac{L \times [0,\infty)}{L \times \{0\}}$$

- manifold with conical L-singularities

M : smooth manifold away from $\Sigma \subseteq M$ discrete

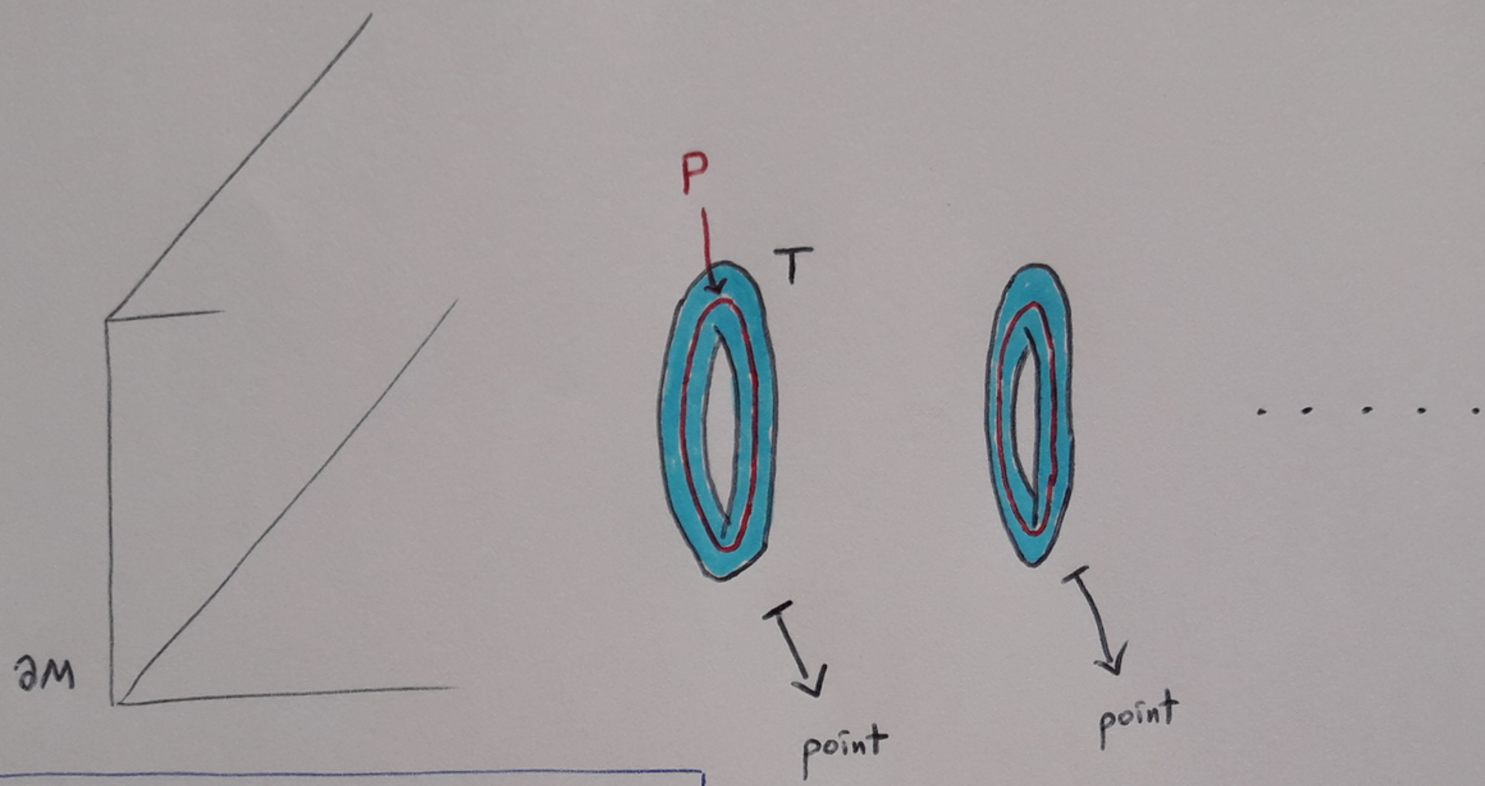
$\sigma \in \Sigma \rightsquigarrow$ nbhd $U_\sigma \cong \mathrm{cone}(L)$

- E.g. graph with uniform valency $\longrightarrow L =$ finite set
- E.g. $\mathbb{R}^3 \supseteq$ link $\rightsquigarrow$ collapse each component to a point $\longrightarrow L = S^1 \times S^1$

- $\mathrm{Diff}^L_G(M) \leqslant \mathrm{Homeo}(M)$
  - $\Sigma \longmapsto \Sigma$
  - diffeo on $M \setminus \Sigma$
  - near $\sigma \in \Sigma$: looks like $\mathrm{cone}(g)$ for $g \in G$

Setup

M_n =



$\boxed{\text{manifold with conical } \partial T\text{-singularities}}$

- Corollary (P.)

if
- $\dim(P) \leq \frac{1}{2}(\dim(M) - 3)$
- conditions on G as before

then $H_* \mathrm{Diff}^{\partial T}_G(M_n)$ stabilises for $n \geq 2$ *

Thank you for your attention !

slides available at mdp.ac/talks.html