1. Configuration spaces with non-local structure.

— Homological stability for oriented configuration spaces; a twisted group-completion theorem; stable homology of oriented configuration spaces. —

For a space M, the nth ordered configuration space $\widetilde{C}_n(M)$ is defined to be the subspace of M^n consisting of all n-tuples of pairwise distinct points in M. The symmetric group $\mathfrak{S}_n$ acts on this space, and we define

$$C_n(M) \;=\; \widetilde{C}_n(M)/\mathfrak{S}_n \qquad \text{and} \qquad C_n^+(M) \;=\; \widetilde{C}_n(M)/A_n,$$

where $A_n < \mathfrak{S}_n$ is the alternating group. These are called, respectively, the *unordered* configuration space on M and the *oriented* configuration space on M. It is a classical result, going back to McDuff [McD75] and Segal [Seg73, Seg79], that the sequence $C_n(M)$, when M is a connected, open manifold, is *homologically stable*. This means that, for each degree i, there are isomorphisms $H_i(C_n(M)) \cong H_i(C_{n+1}(M))$ once n is sufficiently large (depending on i).

Aside. The condition that M is connected is clearly necessary, as one can see by considering H_0. On the other hand, the condition that M is open is not obviously necessary, and the situation in this case is much more subtle. This is the subject of another part of my previous work, see §3.

The ordered configuration spaces $\widetilde{C}_n(M)$ are not homologically stable: for example, the first homology of $\widetilde{C}_n(\mathbb{R}^2)$ is the abelianisation of the pure braid group, which is $\mathbb{Z}^{\binom{n}{2}}$. This raises the question of whether there is an intermediate covering space between $\widetilde{C}_n(M)$ and $C_n(M)$ for which homological stability still holds. I proved in [Pal13b] that the answer is positive for the oriented configuration spaces $C_n^+(M)$, which doubly cover the unordered configuration spaces $C_n(M)$:

Theorem ([Pal13b]) *The natural stabilisation map $C_n^+(M) \to C_{n+1}^+(M)$ induces isomorphisms on homology in degrees $* \leqslant \frac{n-5}{3}$ and surjections in degrees $* \leqslant \frac{n-2}{3}$.*

The theorem also holds more generally for *labelled configuration spaces*, where each point is equipped with a "label" in some fixed, path-connected parameter space X. The so-called *slope* of the stability range here is $\frac{1}{3}$ (and one may calculate explicitly for certain M to see that this is sharp), in contrast to the slope of $\frac{1}{2}$ that holds for the unordered configuration spaces.

Note. The integral homology of $C_n^+(M)$ may be interpreted as a certain twisted homology group $H_*(C_n(M); \mathbb{Z}[\mathbb{Z}/2])$, where $\pi_1(C_n(M))$ acts on the group ring of $\mathbb{Z}/2$ via the natural projection $\pi_1(C_n(M)) \to \mathfrak{S}_n$ followed by the sign homomorphism. This is an example of an *abelian twisted coefficient system* on $C_n(M)$, and is a precursor (in a special case) of the notion of *abelian homological stability*, which has been developed more recently by Randal-Williams and Wahl [RW17] and Krannich [Kra17]. See §2 for more twisted homological stability results.

This result leads to the question of whether one can identify the *stable homology* of the sequence $\{C_n^+(M) \mid n \in \mathbb{N}\}$, in other words the colimit $\lim_{n\to\infty} H_*(C_n^+(M))$, in terms of other well-understood spaces. In joint work with **Jeremy Miller**, we answered this question positively by lifting the classical *scanning map* [Seg73, McD75] to a homology equivalence between appropriate covering spaces.

- First, in our paper [MP15a], we generalise the McDuff-Segal *group-completion theorem* [MS76] as well as McDuff's *homology fibration criterion* [McD75, §5] to the setting of homology with twisted coefficients (more precisely to a setting where we consider homology with respect to all twisted coefficient systems in a fixed class $\mathfrak{C}$ that is closed under pullbacks).

- Using these tools, we proved in [MP15b] a new kind of scanning result, lifting the classical scanning map to covering spaces and showing that it remains a homology equivalence after

doing so. This identifies the stable homology of oriented configuration spaces on M with the homology of an explicit double cover of the section space of a certain bundle over M:

Theorem ([MP15b]) *Writing $\dot{T}M \to M$ for the fibrewise one-point compactified tangent bundle of M and $\Gamma_c(\dot{T}M \to M)_\circ$ for its space of compactly-supported sections of degree zero, we have:*

$$\lim_{n\to\infty} H_*(C_n^+(M)) \;\cong\; H_*(\widetilde{\Gamma_c}(\dot{T}M \to M)_\circ), \tag{1}$$

for a certain explicit double cover $\widetilde{\Gamma_c}(\dot{T}M \to M)_\circ \longrightarrow \Gamma_c(\dot{T}M \to M)_\circ$.

This double cover may be defined as the connected covering space of $\Gamma_c = \Gamma_c(\dot{T}M \to M)_\circ$ corresponding to the projection

$$\pi_1(\Gamma_c) \to H_1(\Gamma_c) \cong H_1(C_2(M)) \to H_1(C_2(\mathbb{R}^\infty)) \cong \mathbb{Z}/(2).$$

Here, the isomorphism $H_1(\Gamma_c) \cong H_1(C_2(M))$ arises from homological stability and the identification of the stable homology for *unordered* configuration spaces, and the map $H_1(C_2(M)) \to H_1(C_2(\mathbb{R}^\infty))$ is induced by any embedding $M \hookrightarrow \mathbb{R}^\infty$. For example, when $M = \mathbb{R}^\infty$ the right-hand side of (1) is the universal cover of (one component of) the infinite loopspace $\Omega^\infty S^\infty = Q\mathbb{S}^0$. When $M = \mathbb{R}^2$ it is the unique connected double cover of $\Omega^2 S^3$.

2. Twisted homological stability.

— Twisted homological stability for configuration spaces; different notions of the degree of twisted coefficient systems. —

As well as the notion of abelian homological stability mentioned in the previous section, another sense in which a sequence of spaces (or groups) can satisfy *twisted homological stability* is with respect to a so-called *polynomial twisted coefficient system*. This consists of a choice of local coefficient system on each space X_n, together with additional morphisms of local coefficient systems between them, organised into a functor $\mathcal{C} \to \mathsf{Ab}$, where the automorphism groups of $\mathcal{C}$ are the fundamental groups $\pi_1(X_n)$. In addition, the *degree* of this functor (defined using extra structure on $\mathcal{C}$) is required to be finite.

Many families of groups $\boldsymbol{G} = \{G_n\}$ are known to be homologically stable in this sense (where we set $X_n = BG_n$ in the above paragraph), for appropriately-defined categories $\mathcal{C}_{\boldsymbol{G}}$, for example the symmetric groups $\mathfrak{S}_n$, braid groups β_n, general linear groups, automorphism groups of free groups $\mathrm{Aut}(F_n)$ and mapping class groups of surfaces and of 3-manifolds.[1] In [Pal13a], I proved the first such result for a sequence of *spaces*, namely the unordered configuration spaces $C_n(M)$ on any connected, open manifold M. Note that, when $\dim(M) \geqslant 3$, these spaces are not aspherical (in contrast to the case of surfaces), so this does not reduce to a statement about the homology of their fundamental groups. When $\dim(M) = 2$, these configuration spaces are aspherical, so the result may be thought of as twisted homological stability for their fundamental groups, the surface braid groups $B_n(M) = \pi_1(C_n(M))$.

Theorem ([Pal13a]) *Let M be an open, connected manifold and let T be a twisted coefficient system for $\{C_n(M)\}$. This includes in particular the data of a local coefficient system T_n for each $C_n(M)$, as well as a homomorphism*

$$H_*(C_n(M); T_n) \longrightarrow H_*(C_{n+1}(M); T_{n+1}).$$

This homomorphism is split-injective in all degrees, and, if T has finite degree d, it is an isomorphism in the range $* \leqslant \frac{n-d}{2}$.

This theorem also generalises to configuration spaces with labels in any path-connected space X, as in the previous section.

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[1] Symmetric groups: [Bet02]; braid groups: [CF13, RW17, Pal13a]; general linear groups: [Dwy80, Kal80]; automorphism groups of free groups: [RW17]; mapping class groups of surfaces: [Iva93, CM09, Bol12, RW17]; mapping class groups of 3-manifolds: [RW17]. Note that these are references for the proofs of *twisted* homological stability; in each case, homological stability with untwisted coefficients was known earlier.

Twisted coefficient systems. In connection with these results, I have also explored in more depth [Pal17] the notion of *twisted coefficient system* (a.k.a. finite-degree or polynomial functor), and in particular the *degree* of a twisted coefficient system. The main results of [Pal17] are:

- A comparison and unification of various different notions of "finite-degree" functor $\mathcal{C} \to \mathcal{A}$, where $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is a category with various kinds of additional structure.
- The development of a functorial construction of (*injective* or *partial*) *braid categories*, which were used in [Pal13a] as the domain of definition of twisted coefficient systems, and of finite-degree functors on such braid categories.

3. The case of closed manifolds.

— Stability phenomena for configuration spaces on closed manifolds. —

When M is closed, homological stability for the unordered configuration spaces $C_n(M)$ is not true in general, for example one may calculate that $H_1(C_n(S^2); \mathbb{Z}) \cong \mathbb{Z}/(2n-2)$, which does not stabilise as $n \to \infty$. Moreover, the stabilisation maps mentioned in the theorem in §1 do not exist, since these depend on adding a new configuration point in M "near infinity". In joint work with **Federico Cantero** [CP15], we prove three main results which show that the homology of configuration spaces on closed manifolds exhibits a large amount of stability despite these issues.

(1) When the Euler characteristic of M is zero, we construct *replication maps* $C_n(M) \to C_{\lambda n}(M)$ for any integer $\lambda \geqslant 2$, and prove that they induce homological stability after inverting λ:

Theorem ([CP15]) *These maps induce isomorphisms on $H_i(-, \mathbb{Z}[\frac{1}{\lambda}])$ in the range $2i \leqslant \lambda$.*

Note. A construction related to our replication maps has also been used in the paper [Ber+06], in which they use something similar to a replication map in §3 to build a crossed simplicial group out of the configuration spaces on any given manifold M that admits a non-vanishing vector field.

(2) When the manifold M is odd-dimensional, the configuration spaces $C_n(M)$ do in fact satisfy homological stability after inverting 2.

Theorem ([CP15]) *When $\dim(M)$ is odd, there are isomorphisms*

$$H_i(C_n(M); \mathbb{Z}[\tfrac{1}{2}]) \cong H_i(C_{n+1}(M); \mathbb{Z}[\tfrac{1}{2}]) \qquad and \qquad H_i(C_n(M); \mathbb{Z}) \cong H_i(C_{n+2}(M); \mathbb{Z})$$

in the range $2i \leqslant n$, induced by a zigzag of maps.

This strengthens a result of [BM14].

(3) When the manifold M is even-dimensional, and $\mathbb{F}$ is a field of characteristic 0 or 2, it is known by the work of many people [BCT89, ML88, Chu12, Ran13, BM14, Knu17] that homological stability holds for $C_n(M)$ with coefficients in $\mathbb{F}$, even when M is closed. When $\mathbb{F}$ has odd characteristic p, however, this is false, as one can see from the example of $M = S^2$ mentioned above. In fact:

$$H_1(C_n(S^2); \mathbb{F}) \cong \begin{cases} \mathbb{F} & p \mid n-1 \\ 0 & p \nmid n-1 \end{cases} \quad \text{for } n \geqslant 2.$$

From this example we see that the first homology of $C_n(S^2)$ is not stable, but it is p-periodic and takes on only 2 different values. Our third result is that this phenomenon holds in general, when the Euler characteristic χ of M is non-zero. Write $a = \nu_p(\chi)$ for the p-adic valuation of χ, in other words $\chi = p^a b$ with b coprime to p.

Theorem ([CP15]) *Suppose that $\dim(M)$ is even. For each fixed i, the sequence*

$$H_i(C_n(M); \mathbb{F}) \quad for \ n \geqslant 2i \tag{2}$$

is p^{a+1}-periodic and takes on at most $a + 2$ values. Moreover, if $\chi \equiv 1 \bmod p$ then the above sequence is 1-periodic, i.e. homological stability holds with coefficients in $\mathbb{F}$.

The p^{a+1}-periodicity result is similar to a theorem of [Nag15], although his estimate of the period is different, namely a power of p depending on i rather than on χ. This result was later improved by [KM16] to p-periodicity, independent of i or χ. Combining this with (a slightly more precise statement of) our result, a corollary is that in fact the sequence (2) above takes on only two different values.

4. Moduli spaces of disconnected submanifolds.

— Moduli spaces of disconnected submanifolds; symmetric diffeomorphism groups; manifolds with conical singularities; partitioned braid groups. —

Instead of configurations of points in M (closed 0-dimensional submanifolds), one may consider configurations of closed submanifolds of M of higher dimension, which are diffeomorphic to the disjoint union of finitely many copies of a fixed ("model") manifold L. In the recent preprint [Pal18a], I proved that *moduli spaces of disconnected submanifolds* of this kind are also homologically stable as the number of components goes to infinity, just as in the classical setting of points [Seg73, McD75, Seg79] – under a certain hypothesis on the relative dimensions of the manifolds involved.

Let $\bar{M}$ be a connected manifold with non-empty boundary and of dimension at least 2, and denote its interior by M. Also fix a closed manifold L and an embedding $\iota_0 \colon L \hookrightarrow \partial\bar{M}$. Choose a self-embedding $e \colon \bar{M} \hookrightarrow \bar{M}$ which is isotopic to the identity and such that $e(\iota_0(L))$ is contained in the interior $M \subset \bar{M}$. We then obtain a sequence of pairwise-disjoint embeddings of L into M by defining $\iota_n := e^n \circ \iota_0$ for $n \geqslant 0$.

Let $nL = \{1, \ldots, n\} \times L$ and write $\iota_{1,\ldots,n} \colon nL \hookrightarrow M$ for the embedding $(i, x) \mapsto \iota_i(x)$.

Definition Define $C_{nL}(M)$ to be the path-component of

$$\mathrm{Emb}(nL, M)/\mathrm{Diff}(nL)$$

containing $[\iota_{1,\ldots,n}]$. Here, the embedding space is given the Whitney topology and $C_{nL}(M)$ the quotient topology. There is a natural *stabilisation map*

$$C_{nL}(M) \longrightarrow C_{(n+1)L}(M) \tag{3}$$

defined by adjoining the embedding ι_0 to a given embedding $nL \hookrightarrow M$, to obtain a new embedding $(n+1)L \hookrightarrow \bar{M}$, and then composing with the self-embedding e. In symbols, this may be written as $[\phi] \mapsto [\phi^+]$, where $\phi^+(i, x) = e \circ \phi(i, x)$ for $1 \leqslant i \leqslant n$ and $\phi^+(n+1, x) = \iota_1(x)$.

Theorem ([Pal18a]) *Assume that the dimensions $m = \dim(M)$ and $\ell = \dim(L)$ satisfy*

$$2\ell \leqslant m - 3. \tag{4}$$

Then (3) *induces isomorphisms on homology in degrees $* \leqslant \frac{n-2}{2}$ and surjections in degrees $* \leqslant \frac{n}{2}$.*

This theorem may be extended further:

- There is a more general version of this setting, in which the submanifolds $L \subset M$ are parametrised modulo a subgroup of $\mathrm{Diff}(L)$ and come equipped with labels in some bundle over $\mathrm{Emb}(L, M)$. The theorem proved in [Pal18a] includes this more general setting.
- This setting is compatible with the techniques of §1 above, so we also have homological stability for "oriented" (in the sense of §1) versions of these moduli spaces, in which the submanifolds $L \subset M$ are ordered modulo even permutations.

Applications to diffeomorphism groups. In the sequel [Pal18b], I used homological stability for the moduli spaces $C_{nL}(M)$ (and their more refined versions mentioned in the first point above) to prove homological stability for:

○ Symmetric diffeomorphism groups, with respect to parametrised connected sum.
○ Diffeomorphism groups of manifolds with conical singularities, with respect to the number of singularities.

Definition Given two embeddings $L \hookrightarrow M$ and $L \hookrightarrow Q$ with isomorphic normal bundles, one may cut out a tubular neighbourhood of each embedding and glue the resulting boundaries to obtain the *parametrised connected sum* $M \sharp_L Q$. If L is a point this corresponds to the ordinary connected sum of M and Q. Other examples include the following.

- If $L \hookrightarrow Q$ is the canonical embedding $S^k \hookrightarrow S^m$, where $k \leqslant m = \dim(M)$, then $M \sharp_L Q$ is the result of a *k-surgery* on M.
- If $L \hookrightarrow Q$ is the embedding $S^1 \hookrightarrow \mathbf{T} \hookrightarrow \mathbf{T} \cup_{p/q} \mathbf{T} = L(p,q)$, where $\mathbf{T} = D^2 \times S^1$ denotes the solid torus, then $M \sharp_L Q$ is the result of a Dehn surgery of slope p/q on the 3-manifold M.

If we now iterate the operation $-\sharp_L Q$ many times, using a different copy of Q and a disjoint embedding $L \hookrightarrow M$ each time (but always using the same copy of M), we obtain a sequence

$$M \qquad M\sharp_L Q \qquad M\sharp_L Q\sharp_L Q \qquad M\sharp_L Q\sharp_L Q\sharp_L Q \qquad \cdots \qquad (5)$$

of manifolds, which we abbreviate to $M\sharp_L^n Q$, the *nth iterated parametrised connected sum*.

A diffeomorphism of $M\sharp_L^n Q$ is called *symmetric* if it fixes the boundary of M and preserves the decomposition of $M\sharp_L^n Q$ into pieces of the form $M \smallsetminus n\mathcal{T}(L)$ and $Q \smallsetminus \mathcal{T}(L)$, where $\mathcal{T}(L)$ is a tubular neighbourhood of L in M or Q. The corresponding subgroup

$$\Sigma\mathrm{Diff}(M\sharp_L^n Q) \ \leqslant \ \mathrm{Diff}(M\sharp_L^n Q)$$

is called the *symmetric diffeomorphism group* of $M\sharp_L^n Q$.[2]

Theorem ([Pal18b]) *If M is connected and has non-empty boundary and $\dim(L) \leqslant \frac{1}{2}(\dim(M)-3)$, the sequence*

$$\cdots \longrightarrow B\Sigma\mathrm{Diff}(M\sharp_L^n Q) \longrightarrow B\Sigma\mathrm{Diff}(M\sharp_L^{n+1} Q) \longrightarrow \cdots$$

of (classifying spaces of) symmetric diffeomorphism groups is homologically stable.

This generalises a result of Tillmann [Til16], which corresponds to the case $L = \mathrm{point}$ (i.e. the usual connected sum operation).

Informal definition Fix an $(m-1)$-dimensional manifold T. Let $\mathrm{cone}(T) = (T \times [0, \infty))/(T \times \{0\})$ be the open cone on T. An m-dimensional *manifold with conical T-singularities* is a space M that is locally homeomorphic to $\mathrm{cone}(T)$, together with a smooth atlas on the subset $M_{mfd} \subseteq M$ of locally Euclidean points of M. A *diffeomorphism* of M is a homeomorphism $M \to M$ that restricts to a diffeomorphism $M_{mfd} \to M_{mfd}$ and is of the form $\mathrm{cone}(\varphi)$ for some diffeomorphism $\varphi \colon T \to T$ near each point of the discrete subset $M \smallsetminus M_{mfd} \subseteq M$. These form a subgroup

$$\mathrm{Diff}^T(M) \ \leqslant \ \mathrm{Homeo}(M).$$

For example, we may construct a manifold with conical singularities by collapsing a tubular neighbourhood $\mathcal{T}(L)$ of any submanifold $L \subset M$. The quotient $M_L = M/\mathcal{T}(L)$ is then a manifold with a single conical $\partial\mathcal{T}(L)$-singularity. In particular, using the setting at the beginning of this subsection, we may collapse a tubular neighbourhood of each submanifold $\iota_i(L) \subset M$ for $1 \leqslant i \leqslant n$, to obtain a manifold $M_{n \cdot L}$ with (precisely n) conical $\partial\mathcal{T}(L)$-singularities.

Theorem ([Pal18b]) *If M is connected and has non-empty boundary and $\dim(L) \leqslant \frac{1}{2}(\dim(M)-3)$, the sequence of classifying spaces $B\mathrm{Diff}^{\partial\mathcal{T}(L)}(M_{n \cdot L})$ is homologically stable.*

Special values of ℓ and m. The condition (4) on the relative dimensions of L and M excludes some interesting special cases of the moduli space $C_{nL}(M)$.

One such special case is $\ell = 1$, $m = 3$, in other words, moduli spaces of links in a 3-manifold. If one considers moduli spaces of *unlinks* in a 3-manifold M, then this is known to be homologically stable as the number of components of the unlink goes to infinity, by a result of Kupers [Kup13]. However, the techniques of Kupers do not generalise to moduli spaces of non-trivial links (even if the components are pairwise unlinked), so this question is open for general links.

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[2] A mild technical condition has been elided from the definition of symmetric diffeomorphism group and in the statement of the theorem below, in order to simplify the discussion.

Another special case is $\ell = 0$, $m = 2$. If L is a point, this corresponds to configuration spaces of points on a surface M, for which homological stability is known classically. However, the only assumption that we have made about L is that it is closed, not necessarily connected, so we could also take $L = \{1, \ldots, \xi\}$ for any positive integer ξ. The moduli space $C_{nL}(M)$ is then the covering space of $C_{n\xi}(M)$ with

$$p_\xi(n) = \frac{(n\xi)!}{n! \, (\xi!)^n}$$

sheets, whose fibres correspond to all ways of partitioning the $n\xi$ points of a configuration into n subsets of size ξ. Since configuration spaces on M are aspherical (M is a connected surface with non-empty boundary), this is equivalent to studying the corresponding index-$p_\xi(n)$ subgroup of the surface braid group $B_{n\xi}(M)$, called the *partitioned surface braid group* $B_{\xi|n}(M)$, consisting of all braids that preserve a given partition $n\xi = \xi + \xi + \cdots + \xi$ of their endpoints. In joint work with **TriThang Tran**, we have shown that homological stability holds also in this special case, corresponding to $(\ell, m) = (0, 2)$.

Theorem ([PT14]) *Let M be a connected surface with non-empty boundary. Then the sequence of partitioned surface braid groups $B_{\xi|n}(M)$ is homologically stable as $n \to \infty$, for any fixed $\xi \geqslant 1$.*

This recovers the classical case of homological stability for configuration spaces of points when $\xi = 1$, and is new for $\xi \geqslant 2$.

5. Higher crossing diagrams in knot theory.

— Reidemeister moves for triple-crossing link diagrams; relations between different n-crossing numbers for links. —

The last theme of my previous work is in knot theory (which one may think of as the study of π_0 of the moduli space of 1-dimensional submanifolds of the 3-sphere), and more precisely with the representation of links by diagrams in the plane (or 2-sphere). A classical link diagram is an immersion of a 1-manifold into the plane, which is an embedding except at a finite number of double points, where the 1-manifold must intersect itself transversely, together with additional data at each intersection point specifying which strand passes "over" the other at that point. For a given link, such a diagram is unique up to ambient isotopy and the well-known *Reidemeister moves*.

In joint work with **Colin Adams** and **Jim Hoste** [AHP17], we study instead *triple-crossing diagrams*, which consist of an immersed 1-manifold in the plane, which is an embedding except at a finite number of points, at which *exactly three* strands must intersect transversely (plus additional data at each intersection point specifying which strands pass "over" others). We introduce an analogue of the Reidemeister moves for such diagrams, consisting of:

○ Analogues of the (classical) I- and II-moves, which may be thought of as surgeries supported on a small subdisc of a diagram.
○ The *trivial pass move*, which consists of cutting a strand and re-attaching it through another part of the diagram without introducing any new crossings. This may be thought of as a surgery on an annular neighbourhood of a diagram.
○ Two families of moves, called the *band moves* and *basepoints moves*, which each consist of a surgery supported on a pair of disjoint subdiscs of the diagram.

Definition For a link L in S^3, a *maximal nonsplit sublink* of L is a sublink L_1 of L such that (a) there exists an embedded 2-sphere in $S^3 \smallsetminus L$ separating L_1 and $L \smallsetminus L_1$, and (b) there does *not* exist any embedded 2-sphere in $S^3 \smallsetminus L_1$ separating L_1 into two smaller sublinks.

A *relative orientation* of L is an orientation of L modulo orientation-reversal of each maximal nonsplit sublink. Equivalently, this is a choice, for each maximal nonsplit sublink L_1 of L, of an orientation of L_1 modulo complete reversal (reversing every component of L_1 simultaneously).

Theorem ([AHP17]) *A triple-crossing diagram determines a relatively oriented link. Any two triple-crossing diagrams representing the same relatively oriented link differ by a finite sequence of I-moves, II-moves, trivial pass moves, band moves, basepoints moves and ambient isotopy.*

The notion of triple-crossing diagram may be generalised to any integer $n \geqslant 2$, with $n = 2$ corresponding to the classical notion of link diagram. The *n-crossing number* $c_n(L)$ of a link L is then the smallest number of crossings among all n-crossing diagrams of that link. One can ask how the sequence $\{c_n(L) \mid n \in \mathbb{N}\}$ behaves for each L, and which relations between the crossing numbers hold for all links L (or for all but finitely many links L). For example, it is not hard to show that $c_n(L) \geqslant c_{n+2}(L)$ for all n and L, and this inequality is known be strict for $n = 2$. We prove that, with a few small exceptions, it is also strict for $n = 3$.

Theorem ([AHP17]) *Let L be a non-split link that is neither the unlink nor the Hopf link. Then*

$$c_3(L) > c_5(L). \tag{6}$$

Note that this inequality is clearly false for the unlink and the Hopf link: for these links, $c_3(L)$ and $c_5(L)$ are both equal to 0 and both equal to 1, respectively.

Also note that, if L can be written as the union of maximal nonsplit sublinks $L_1, \ldots, L_k$, then $c_n(L) = c_n(L_1) + \cdots + c_n(L_k)$. Thus the strict inequality (6) holds whenever L has at least one maximal nonsplit sublink that is nontrivial and not the Hopf link.

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