Reidemeister moves for triple-crossing link diagrams

Seminar talk at IMAR // Martin Palmer-Anghel // 21 December 2018

Reidemeister moves:

A classical link diagram is an immersion of a 1-manifold into the plane, which is an embedding except at a finite number of double points, where it must intersect itself transversely, together with additional data at each intersection point specifying which strand passes "over" the other at that point. For a given link, such a diagram is unique up to ambient isotopy and the classical *Reidemeister moves* – a finite collection of local modifications of classical link diagrams.

The first part of this talk will be about *triple-crossing diagrams*, which consist of an immersed 1-manifold in the plane that is an embedding except at a finite number of points, at which *exactly three* strands must intersect transversely (plus additional data at each intersection point specifying which strands pass "over" others). These were introduced in [1], where it was also shown that every link may be represented by at least one triple-crossing diagram.

A natural question is: how unique is this representation? In other words: is there an analogous collection of local modifications of triple-crossing diagrams playing the role of the classical Reidemeister moves? I will describe a candidate collection of such local modifications:

- Analogues of the (classical) I- and II-moves, which may be thought of as surgeries supported on a small subdisc of a diagram.
- The *trivial pass move*, which consists of cutting a strand and re-attaching it through another part of the diagram without introducing any new crossings. This may be thought of as a surgery on an annular neighbourhood of a diagram. (For classical link diagrams, this move is implied by the classical I-, II- and III-moves.)
- Two new families of moves, called the *band moves* and *basepoints moves*, which each consist of a surgery supported on a pair of disjoint subdiscs of the diagram.

The aim of the first part of this talk will then be to explain the ideas of a proof that these moves do indeed play the role of the classical Reidemeister moves for triple-crossing diagrams:

Theorem. [2] (Joint with C. Adams and J. Hoste) *Every triple-crossing diagram determines a relatively oriented link. If two triple-crossing diagrams represent the same relatively oriented link, then they differ by a finite sequence of the above moves and ambient isotopy.*

(A *relative orientation* of a link L is a choice of orientation of each component of L, modulo orientation-reversal of each maximal non-split sublink. So, for example, the Hopf link has two possible relative orientations, whereas a relatively oriented *knot* is just an *unoriented* knot.)

Triple-crossing numbers of links:

In the second part, I will discuss the *n-crossing number* $c_n(L)$ of a link L, which is the smallest number of crossings amongst all n-crossing diagrams representing L. The *Monotonicity Conjecture* says that the sequence

$$\{c_n(L) \mid n \in \mathbb{N}\}$$

is non-increasing for any link L. What one can show easily is that $c_n(L) \geqslant c_{n+2}(L)$ for all n and (via an argument originally due to V. Jones) that $c_n(L) \geqslant c_{2n}(L)$ for all n. I will explain these arguments, and also prove two stronger facts about $c_3(L)$:

$$c_2(L) > c_3(L) > c_5(L),$$

where the last strict inequality holds unless every split summand of L is trivial or the Hopf link.

References:

[1] Colin Adams, *Triple crossing number of knots and links*, J. Knot Theory Ramifications 22.2, 1350006, 2013.

[2] Colin Adams, Jim Hoste, Martin Palmer, *Triple-crossing number and moves on triple-crossing link diagrams*, arXiv:1706.09333, 2017. To appear in J. Knot Theory Ramifications.