Stability and stable homology for moduli spaces of disconnected submanifolds

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1. Moduli spaces of disconnected submanifolds
2. Stability $(n \to \infty)$
3. Stable homology $(n = \infty)$

[4. Diffeomorphism groups of manifolds with singularities] → if time permits

1.

Def Fix $\underset{\substack{\text{closed}\\ \text{connected}}}{L} \xhookrightarrow{e} \partial \bar{M}$, $\bar{M}$ conn. mfld, $M = \mathrm{int}(\bar{M})$.

$$C_{nL}(M) = \text{path-comp}^t \text{ of } \mathrm{Emb}(\textstyle\coprod_n L, M) / \mathrm{Diff}(\coprod_n L) \quad \text{containing } [ne]$$

n parallel copies of e in a collar nbhd of $\partial\bar{M}$

$\pi_1 C_{nL}(M)$ is the corresponding motion group

Eg $L = \text{point} \longrightarrow$ configuration spaces, braid groups (classical when $M = \mathbb{R}^2$)

$L = S^1 \hookrightarrow \partial B^3 \longrightarrow$ space of n-comp^t unlinks, LB_n

extended version including 180° twists.

Aim Understand $H_* C_{nL}(M)$.

Note This is not usually the same as $H_* \pi_1 C_{nL}(M)$

↳ **Eg** if $L = \text{point}$ & $M = \text{aspherical surface}$, they are the same (Fadell–Neuwirth '62)

if $L = S^1 \hookrightarrow \partial B^3$, they are not

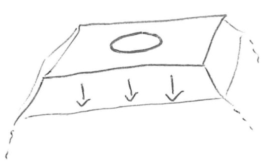
(i) $H_* C_{nS^1}(\mathbb{R}^3) = 0$ for $* > 6n$ [Brendle–Hatcher]

(ii) $H_* LB_n \neq 0$ for arbitrarily large $*$ (LB_n contains torsion).

2.

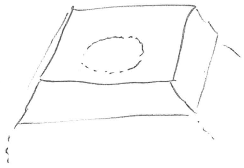
Consider $n \longrightarrow \infty$.

Def $C_{nL}(M) \xrightarrow{\;s\;} C_{(n+1)L}(M)$



$\longmapsto$

- adjoin $[e]$ to a config.
- push into the interior of $\overline{M}$



$\overline{M}$

L = point

Thm (McDuff, Segal '70s)

(a) s induces $\cong$ on homology up to degree $n/2$

(b) Construct "computable" spaces $X(M)$ such that $\lim_{n\to\infty} H_* C_n(M) \cong H_* X(M)$.

eg $X(\mathbb{R}^d) = (\Omega^d S^d)_\bullet$ ← one path-comp^t

$\Omega^d(-) = Map_*(S^d, -)$.

dim(L) > 0

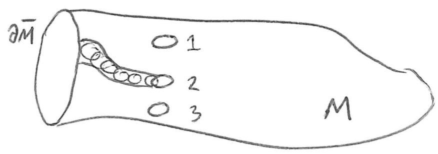
Thm (Kupers '13) For the example $C_{nS^1}(\mathbb{R}^3)$, s induces $\cong$ on homology up to degree $n/2$.

(P. '18) Whenever $\dim(L) \leq \frac{1}{2}(\dim(M)-3)$, ——————— " ———————.

Side note:
- hom stab^y for LB_n : Hatcher-Wahl '10
- calculation of H_* of LB_n : Griffin '13

Idea of proof.

- Build simplicial cx of "ways to undo the map s".
- Prove that this is highly-connected (π_* vanishes for $* \leq n/2$)
- [Machine due to Quillen + many others] $\leadsto$ homological stability.
- I'll describe a "toy model" of the simplicial cx in our case & prove that it's contractible.
- Fix $e_0 : L \times \{1,\dots,n\} \hookrightarrow M$



- Vertices $= \left\{ e : L \times [0,1] \hookrightarrow \overline{M} \;\middle|\; \begin{array}{l} e(L\times\{0\}) \subseteq \partial\overline{M} \\ e(L\times\{1\}) = e_0(L\times\{k\}),\ k=1\dots n \end{array} \right\}$.

a set of vertices spans a simplex $\iff$ disjoint except at $L\times\{1\}$. $\leadsto X$

- Transversality + $\dim(L\times[0,1]) < \frac{1}{2}\dim(M)$
  $\Rightarrow$ for any finite set of vertices $e^1 \dots e^s$, X contains $\mathrm{Cone}(\mathrm{span}(e^1 \dots e^s))$
- Hence X is contractible. $\square$

3.

What about $n=\infty$? Let $C_{\infty L}(M)$ = direct limit of $C_{nL}(M)$ along s.

<u>L = point</u> (McDuff, Segal)

$$C_\infty(\mathbb{R}^d) \xrightarrow{H_* \cong} \underbrace{\Omega^d_\circ}_{\substack{\text{one path-comp}^t\\ \text{of } d\text{-fold loopspace}}} \underbrace{Z(\mathbb{R}^d)}_{\coprod_n C_n(\mathbb{R}^d)/\sim}$$

$c \sim c'$ iff $c \cap (0,1)^d = c' \cap (0,1)^d$

$$Z(\mathbb{R}^d) \simeq (\mathbb{R}^d)^+ = S^d.$$

<u>Guess</u>

$$C_{\infty L}(\mathbb{R}^d) \xrightarrow{H_* \cong ?} \Omega^d_\circ \underbrace{Z_L(\mathbb{R}^d)}_{\coprod_n C_{nL}(\mathbb{R}^d)/\sim}$$

$$Z_L(\mathbb{R}^d) \simeq \underbrace{\mathrm{Aff}_\ell(\mathbb{R}^d)^+}_{\text{1-pt compact}^n}$$

$\ell = \dim(L)$

Aff_ℓ = affine ℓ-planes in $\mathbb{R}^d$

<u>Counterexample</u> $L = S^1 \hookrightarrow \partial B^3$

- [Brendle–Hatcher] $H_1 C_{nS^1}(\mathbb{R}^3) \cong \underline{(\mathbb{Z}/2)^3}$ $(n \geq 2)$
- $H_1\left(\Omega^3_\circ \mathrm{Aff}_1(\mathbb{R}^3)^+\right)$ is <u>infinite</u> ※.

- generated by σ_1, ρ_1, τ_1
- note: τ_1 and σ_1 have order 2 in LB_n
- ρ_1 has ∞ order in LB_n, but after abelianisation it has order 2 due to the rel^n: $\tau_2 \rho_1 = \sigma_1^{-1} \rho_1^{-1} \sigma_1 \tau_1$ (note that $\tau_1 = \tau_2$ in $(LB_n)^{ab}$)

$\mathrm{Aff}_1(\mathbb{R}^3)^+ = Th(T\mathbb{RP}^2)$

$\tilde{H}_*(Th(T\mathbb{RP}^2)) \underset{\text{Thom}}{\cong} H_{*-2}(\mathbb{RP}^2; \mathcal{O}) \underset{\text{Poincaré}}{\cong} H^{4-*}(\mathbb{RP}^2)$

$\mathbb{Q}$-coeffs: $\begin{cases} \mathbb{Q} & *=4 \\ 0 & *\neq 4 \end{cases}$

$\mathbb{Q}$ Hurewicz theorem $\rightarrow$ this is also $\pi_*(\cdot) \otimes \mathbb{Q}$ for $* \leq 6$.

$H_1 \Omega^3_\circ Th(T\mathbb{RP}^2) \cong \pi_4 Th(T\mathbb{RP}^2) \otimes \mathbb{Q} \cong \mathbb{Q}$.

- Need: $\pi_1 Th(T\mathbb{RP}^2) = 0$. Use SvK:

$$\begin{array}{ccc} \pi_1(ST\mathbb{RP}^2) & \longrightarrow & 0 \\ \Downarrow & & \downarrow \\ \mathbb{Z}/2 \cong \pi_1(\mathbb{RP}^2) & \longrightarrow & \pi_1 Th(T\mathbb{RP}^2) = 0 \end{array}$$

New idea

$$\hat{Z}_L(\mathbb{R}^d) \subseteq Z_L(\mathbb{R}^d)$$

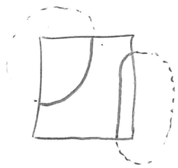
$\quad ||$

subspace of those submflds of $(0,1)^d$ that are disjoint from

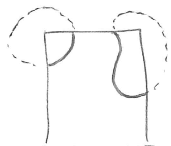
$$\bigcup_{i=1}^{d} (0,1)^{i-1} \times \{t_i\} \times (0,1)^{d-i}$$

for some $t_1, \ldots, t_d \in (0,1)$.

Eg



$\times$



$\checkmark$

Note: Automatic if L = point.

Thm in progress (P. '19)

$$C_{\infty L}(\mathbb{R}^d) \xrightarrow{H_* \cong} \Omega^d_\bullet \hat{Z}_L(\mathbb{R}^d).$$

Rmk $\hat{Z}_L(\mathbb{R}^d) \not\simeq \mathrm{Aff}_\ell(\mathbb{R}^d)^+$ in general.

$\Big($ $\simeq$ if L = point

$\quad\not\simeq$ if $L = S^1 \hookrightarrow \partial B^3$ $\Big)$.

Current work in progress: Hty type of $\hat{Z}_L(\mathbb{R}^d)$ ...?