On homological stability for configuration-section spaces

Joint with Ulrike Tillmann.

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(virtual)
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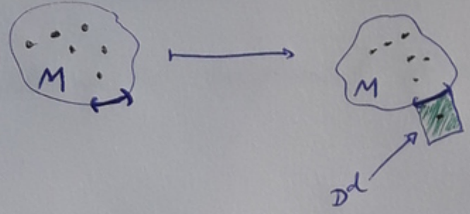
Introduction / motivation

Configuration spaces

$$\underset{\text{manifold}}{M} \rightsquigarrow C_k(M) = \{ (p_1, \dots, p_k) \in M^k \mid p_i \neq p_j \text{ for } i \neq j \} / \Sigma_k$$

→ Stability

if $\partial M \neq \emptyset$ $\rightsquigarrow$ stabilisation maps $C_k(M) \longrightarrow C_{k+1}(M)$



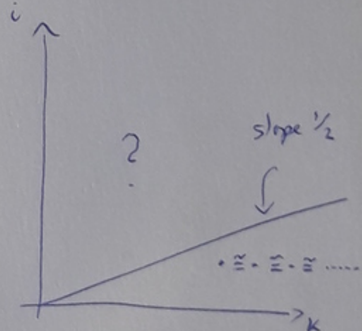
$M \natural D^d \cong M$

THM (McDuff, Segal '70s)

if $\partial M \neq \emptyset$ and M connected

then $H_i C_k(M) \longrightarrow H_i C_{k+1}(M)$

is an isomorphism for $k \geq 2i$

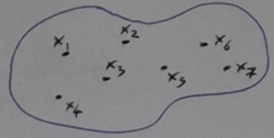


and $\varinjlim_k H_i C_k(M) \cong H_i \underbrace{\Gamma_\cdot(\dot{T}M)}_{\Omega^d S^d \text{ if } M = \mathbb{R}^d}$.

→ What about configurations with labels / parameters?

X parameter space

Def $$C_k(M;X) = \frac{\{(p_1,\dots,p_k)\in M^k \mid p_i \neq p_j \text{ if } i\neq j\}\times X^k}{\Sigma_k}$$



("local")

THM (Randal-Williams '13) The previous theorem generalizes to $C_k(M;X)$ if X is path-connected.

(In particular, $$\varinjlim_k H_i C_k(M;X) \cong H_i \Gamma_c(\dot{T}M \wedge X)$$)

→ What about configurations equipped with non-local data?

↳ e.g. a "field" defined on the complement of the configuration.

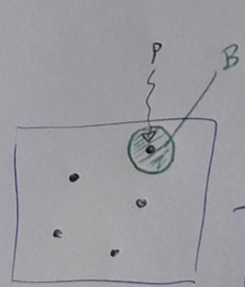
Def (Configuration-mapping spaces) (EVW'12)

M — manifold with $D^{d-1} \subseteq \partial M$

X — based space

$c \subseteq [S^{d-1}, X]$ — "set of allowed charges"

$$CMap_k^c(M,X) = \left\{ \begin{array}{l} z \in C_k(\mathring{M}) \\ f: M \setminus z \longrightarrow X \\ \quad f(D^{d-1}) = * \end{array} \;\middle|\; \begin{array}{l} \forall p \in z \quad \forall B: \\ [f|_{\partial B}] \in c \end{array} \right\}$$



(small subtlety depending on orientability of M)

Physically :

$z \longleftrightarrow$ "particles" in M

$f \longleftrightarrow$ "field"

$[f|_{\partial B}] \longleftrightarrow$ "charge" of $p \in z$

Generalisation :

$$\begin{array}{c} E \\ \downarrow \\ M \end{array}$$ bundle with basepoint in fibre over $0 \in D^{d-1} \subseteq \partial M$

[+ "set of allowed charges" – more technical]

$$\rightsquigarrow C\Gamma^c_k(M,E) = \left\{ \begin{array}{l} z \in C_k(\mathring{M}) \\ \text{section of } E \text{ over } M \setminus z \end{array} \;\middle|\; \cdots \right\}$$

"configuration - section spaces"

Natural fibre sequence :

$$\mathrm{Map}^c_*(M \setminus z, X) \longrightarrow C\mathrm{Map}^c_k(M, X) \xrightarrow{\text{forget the field}} C_k(\mathring{M}) \ni z$$

Examples :

(1) $\begin{array}{c} E = TM \setminus M \\ \downarrow \\ M \end{array} \longrightarrow$ (configuration) + (non-vanishing vector field on the complement)

$\boxed{d \geq 3}$

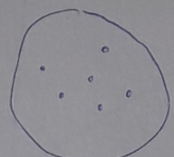
(2) $X = K(G, d-1)$

abelian

$M = D^d$

$\boxed{\text{charges} \longleftrightarrow \text{elements of } G}$

$c \subseteq G$

$\rightsquigarrow$ (configuration) +  $\longrightarrow K(G, d-1)$

non-canonical

$\bigvee_{z} S^{d-1} \xrightarrow{\simeq}$ (configuration), $\quad \bigvee_{z} S^{d-1} \longrightarrow K(G, d-1)$

up to $\simeq$ $\longleftrightarrow$ k-tuple of elements of $c \subseteq G$

$\boxed{d = 2}$

(3) $X = K(G, 1) = BG$

$M = D^2$

$\boxed{\text{charges} \longleftrightarrow \text{conjugacy classes of } G}$

$c \subseteq \mathrm{Conj}(G)$

$\rightsquigarrow$ similar non-canonical description as above, except with

"k-tuple of elements of $\bigcup_{\gamma \in c} \gamma \subseteq G$".

$$\mathrm{CMap}_k^c(D^2, BG) \simeq \mathrm{Hur}^c_{G,k}$$

Hurwitz space

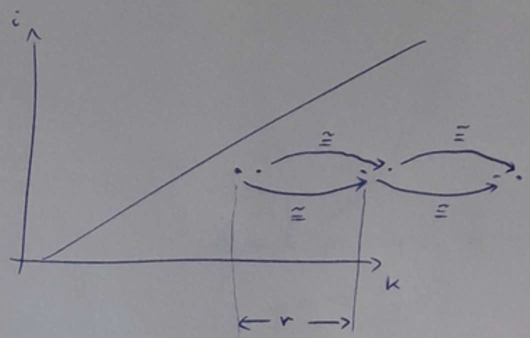
$\{\text{branched } G\text{-coverings } S \to D^2 \text{ with monodromy in } c\} / \cong$

THM ① (EVW '16) if G finite

c is a single conj. class that generates G and is non-splitting

then $H_i(\mathrm{Hur}^c_{G,k};\mathbb{Q}) \cong H_i(\mathrm{Hur}^c_{G,k+r};\mathbb{Q})$

for $k \gg i$ and some r depending only on (G,c).



Coro (EVW'16) Cohen-Lenstra conjecture for extensions of $\mathbb{F}_q(t)$ as $q \longrightarrow \infty$

THM ② (EVW'12) $\varinjlim_k H_i(\mathrm{CMap}^c_k(D^d,X)) \cong H_i(\Omega^{d-1}A'_d(X,c))$

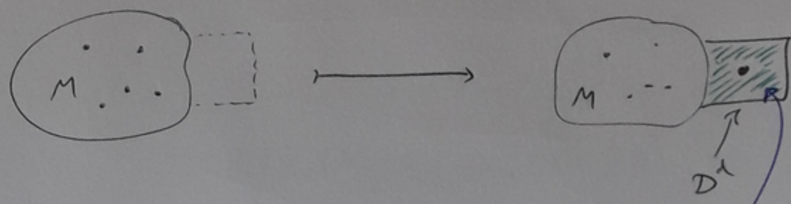
Note: $\coprod_k \mathrm{CMap}^c_k(D^d,X)$ E_{d-1}-algebra

& more generally for M.

(!) stab maps in ① $\neq$ stab maps in ② (in general).

THM (R.-Tillmann '20) Assume M is connected.

stab. maps: $$CMap_k^c(M,X) \xrightarrow{(*)} CMap_{k+1}^c(M,X)$$



depends on choice of

$$\begin{tikzcd} g \in \pi_{d-1}(X) \arrow[d, squiggly] \arrow[d] \\ c \subseteq [S^{d-1}, X] = \pi_{d-1}(X)/\pi_1(X) \end{tikzcd}$$

(a) Suppose:

(†) $c = \{\gamma\}$, γ is a $\pi_1(X)$-orbit in $\pi_{d-1}(X)$ of size 1.

Then $(*)$ induce isomorphism on H_i for all $k \geq 2i$.

(+ gen. for $C\Gamma_k^c(M,E)$)

(b) Extension of EVW's identification of $\varinjlim_k H_i$ for $C\Gamma_k^c(M,E)$.

(c) [in progress] Suppose (†) and

(‡) $\dim(M) \geq 3$ and $\left[\begin{array}{l} \pi_1(M) = 0 \\ \text{or handle-dim}(M) \leq d-2 \end{array}\right]$

Then $(*)$ induces injections on H_i in all degrees.

Rmks
- stab maps
- intersection with EVW
- (c) is false for "oriented configuration spaces"

Ideas of proofs of (a) and (c)

$$\boxed{\mathrm{Map}^c_*(M\setminus k, X) \longrightarrow \mathrm{CMap}^c_k(M,X) \longrightarrow C_k(\mathring{M})}$$

monodromy action

Same SS

$$H_i\Big(C_k(\mathring{M});\, H_j\big(\mathrm{Map}^c_*(M\setminus k, X)\big)\Big) \Rightarrow H_*\big(\mathrm{CMap}^c_k(M,X)\big)$$

$$\pi_1 C_k(\mathring{M}) \overset{\simeq}{\curvearrowright} \mathrm{Map}^c_*(M\setminus k, X)$$

$$\coprod_k \pi_1 C_k(\mathring{M}) \xrightarrow{F} \mathrm{hoTop}$$

$$\mathcal{C}(M)$$

$$\downarrow$$

$$B_\#(M)$$

"braid categories on M"

History:

1980 Dwyer $GL_n(R)$

2002 Betley Σ_k

2014 Randal-Williams–Wahl

[Krannich '17] or [P. '13]

(†)

(a): extend F to $\mathcal{C}(M)$ $\longrightarrow$ polynomiality of $\mathcal{C}(M) \to \mathrm{hoTop} \xrightarrow{H_i} k\text{-Mod}$ $\longrightarrow$ twisted homological stability for $C_k(M)$ (E^2 pages) $\square$

$$\mathcal{C}(M) = \left[\begin{array}{l} \text{gen: } \coprod_k \pi_1 C_k(M) \text{ \& "stab"} \\ \text{rel: } [\text{------}] \end{array}\right]$$

(‡)

(c): explicit formulas for the monodromy action

↳ extend to $B_\#(M)$ $\xrightarrow{[P.'13]}$ split-inj. on E^2-pages ⇢ inj. in the limit.

"combinatorial"

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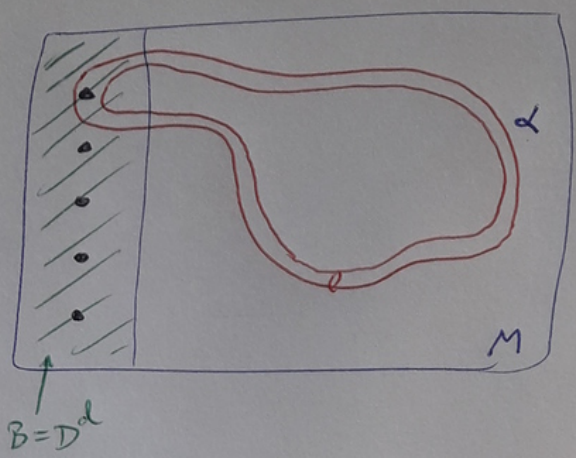
$$\pi_1 C_k(M) \xrightarrow{\simeq} M \cdot k \simeq M \vee \bigvee^{k} S^{d-1}$$

$$\begin{array}{c} \pi_1 C_k(M) \\ \cong \\ \pi_1(M)^k \rtimes \Sigma_k \end{array}$$

<u>Prop</u>. $\sigma \in \Sigma_k$ acts by permutation on the S^{d-1} factors

$\alpha \in \pi_1(M)$



acts on $M \vee S^{d-1}$ by:

$$\begin{array}{c|c} & \begin{array}{cc} M & S^{d-1} \end{array} \\ \begin{array}{c} M \\ S^{d-1} \end{array} & \left(\begin{array}{c|c} id_M & \alpha \circ \mathrm{coll} \\ \hline * & \mathrm{sgn}(\alpha) \end{array}\right) \end{array}$$

description of

$$\begin{tikzcd} M \vee S^{d-1} \arrow[r] & M \vee S^{d-1} \arrow[dl, two heads] \arrow[d, two heads] \\ M & S^{d-1} \end{tikzcd}$$

coll: $S^{d-1} \twoheadrightarrow I \xrightarrow{\alpha} M$

S^{d-1} interval