On motivic cohomological stability for configuration spaces

e-GeMAT, IMAR.

10 April 2020

Joint with Geoffroy Horel

Classical homological stability

M connected, non-compact topological manifold

Def $C_n(M) = \{(x_1 \ldots x_n) \in M^n \mid x_i \neq x_j \text{ for } i \neq j\} / \Sigma_n$

Thm (McDuff, Segal '70s)

$\exists$ stab maps $C_n(M) \longrightarrow C_{n+1}(M)$

that induce isom on (singular) homology in degree i

for $i \leq n/2$

Eg if $M = \mathbb{R}^2$ then $C_n(\mathbb{R}^2) \simeq K(B_n, 1)$

so the braids groups are homologically stable

(This case was proven earlier by Arnol'd)

Question:

if X is a $\boxed{\text{smooth algebraic variety}}$, one can define

analogous configuration varieties $C_n(X)$

Q_1: Are there analogous stab^n map?

Q_2: Does the étale / motivic cohom. of

$C_n(X)$ stabilise along these maps?

side note: $C_n(X)$ is really a scheme.

Background on motivic / Etale cohom

Motivation: X alg variety $/k$ $\qquad k \subseteq \mathbb{C}$ subring

Def The Chow groups of X [Chow, Chevalley]

are

$$CH_i(X) = \frac{\mathbb{Z}\{i\text{-dim subvarieties of } X\}}{\langle (f) \mid f \text{ rational function on an } (i+1)\text{-dim subvar of } X \rangle}$$

divisor of f

and contain a lot of geometric information about X.

- Extended by [Bloch] to higher Chow groups, which coincide with the BM-motivic-homology groups of [Voevodsky]

$$CH_i(X) \cong H^{BM}_{2i,i}(X;\mathbb{Z})$$

- [Voevodsky] also defined motivic cohomology groups

$$H^{i,j}(X;\mathbb{Z})$$

- They are related by Poincaré duality $H^{i,j}(X;\mathbb{Z}) \cong H^{BM}_{2n-i,n-j}(X;\mathbb{Z})$ if X is smooth and n-dim

---

(More generally, with coeffs Λ any ring)

Étale cohomology $H^p_{ét}(-;\Lambda)$

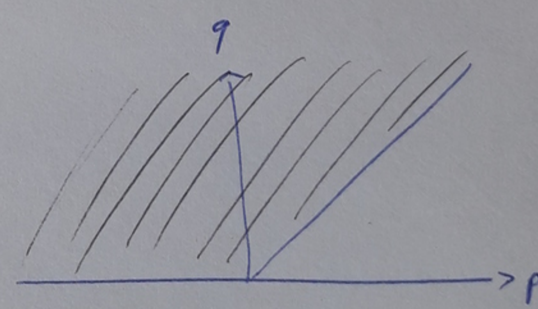
- Defined by Grothendieck to prove the Weil conjectures.

- Voevodsky defined "motivic étale cohomology" $H^{p,q}_{ét}(-;\Lambda)$ analogously to motivic cohomology.

- It recovers étale cohom. when $q=0$ under good conditions ($\Lambda = \mathbb{Q}$ or $\Lambda = \mathbb{Z}/\ell$ and $\frac{1}{\ell} \in k$)

- Heuristically, $H^{pq}_{ét}$ is easier to understand, but H^{pq} contains deeper information.

- They are isomorphic for $q \geq p$ if $\Lambda = \mathbb{Z}/\ell$ and $1/\ell \in k$



[Voevodsky]

"Beilinson-Lichtenbaum conj."

Bird's eye view:

$\begin{pmatrix} k \subseteq \mathbb{C} \\ \Lambda \text{ ring} \end{pmatrix}$

$$\begin{tikzcd}
 & \Lambda\text{-Mod}^{*,*} & & \Lambda\text{-Mod}^{*} \\
 & \mathrm{Mot}_{\acute{e}t} \arrow[u, "H^{*,*}_{\acute{e}t}"] \arrow[drr, bend left, "B"] & & \\
 \mathrm{SmVar}_k \arrow[ur, bend left, "M"] \arrow[dr, bend right, "(-)_{\mathbb{C}}"'] & & & D(\mathrm{Chain}_\Lambda) \arrow[uu, "H^*"'] \\
 & \mathbb{C}\text{-manifolds} \arrow[urr, "C_*(-;\Lambda)"'] & &
\end{tikzcd}$$

Results

X smooth affine alg variety $/ k \subseteq \mathbb{C}$: $X_{\mathbb{C}}$ connected

Thm (Morel - P. '20)

(0) $\exists$ stab maps $M(C_n(X)) \longrightarrow M(C_{n+1}(X))$
that lift the classical $C_n(X_{\mathbb{C}}) \longrightarrow C_{n+1}(X_{\mathbb{C}})$
along B

(1) These induce isom's on $H^{pq}_{\acute{e}t}$ for $p \leq n/2$

(2) If $X = \mathbb{A}^d$, they also induce isom's on $H^{p,q}$ for $p \leq n/2$

Small print: Need to assume that $M(X)$ is "mixed Tate".
This holds e.g. if X = hyperplane complement in $\mathbb{A}^d$.

Plan:

- Detecting isomorphisms
- Construction of motivic stab maps & étale stability
- Pure motives & Postnikov filtration
- Twisted homological stability for Σ_n
- Motivic stability

Detecting isomorphisms

$$\begin{tikzcd}
\Lambda\text{-}\mathrm{Mod}^{\cdot\cdot} & & \Lambda\text{-}\mathrm{Mod}^{*} \\
\mathrm{Mot}_{\text{ét}} \arrow[u, "H^{\cdot\cdot}_{\text{ét}}"] \arrow[rr, "B"] & & D(\mathrm{Chain}_{\Lambda}) \arrow[u, "H^{\cdot}"']
\end{tikzcd}$$

Prop ① if $M_1 \to M_2$ induces isom's on $H^p(B(-))$ $\forall p \leq k$

then ———————— $H^{p,q}_{\text{ét}}(-)$ $\forall p \leq k$

Note — analogue for Mot is <u>false</u>.

Instead — each motive $M \in \mathrm{Mot}$ has a filtration

$$\begin{tikzcd}
w_{\geq s}(M) \arrow[r] \arrow[d] & w_s(M) \\
\vdots \arrow[d] & \\
w_{\geq 1}(M) \arrow[r] \arrow[d] & w_1(M) \\
w_{\geq 0}(M) \arrow[r] \arrow[d] & w_0(M) \\
M &
\end{tikzcd}$$

"strata"

$$\begin{tikzcd}
\Lambda\text{-}\mathrm{Mod}^{\cdot\cdot} & & & \Lambda\text{-}\mathrm{Mod}^{*} \\
\mathrm{Mot} \arrow[u, "H^{\cdot\cdot}"] \arrow[r, "w_s"] & \mathrm{Mot} \arrow[rr, "B"] & & D(\mathrm{Chain}_{\Lambda}) \arrow[u, "H^{\cdot}"']
\end{tikzcd}$$

Prop ②: if $M_1 \to M_2$ induces isom's on $H^p(B(w_s(-)))$ $\forall s \geq 0$ $\forall p \leq k$

then ———————— $H^{p,q}(-)$ $\forall p \leq k$.

<u>Prop</u>:

$$\exists \quad M(C_n(X)) \longrightarrow M(C_{n+1}(X)) \qquad \text{in } Mot \; / \; Mot_{\acute{e}t}$$

recovering the classical stab^n maps.

<u>Idea</u>:

Let $PC_n(X)$ be the <u>ordered</u> config variety
We'll construct Σ_n-equivariant

$$M(PC_n(X)) \longrightarrow M(PC_{n+1}(X))$$

Choose a <u>smooth compact^n</u> $\overline{X}$ of X, so $\overline{X} - X$

is a union of normal-crossing divisors.

Remove all but one (called D) to obtain a smooth

variety Y containing X, such that

$$Y \setminus X = D \quad \text{smooth}.$$

Write $i : D \hookrightarrow Y$. This has a <u>normal

bundle</u> $N(i) \to D$. Write $N_0(i)$ for the

complement of the 0-section.

Let $Z \subseteq PC_{n+1}(Y)$ where first n points lie in X

$\cup$

Z_0 where last point is in D

Then $Z_0 \cong PC_n(X) \times D$

$Z - Z_0 \cong PC_{n+1}(X)$

Note that $N_0(Z_0 \xrightarrow{j} Z) \cong PC_n(X) \times N_0(i)$

__Lemma__ $\exists$ "tubular nbhd embedding"

$$\begin{tikzcd}
M(N_0(i)) \arrow[r] \arrow[d, equal] & M(Z - Z_0) \arrow[d, equal] \\
M(PC_n(X) \times N_0(i)) & M(PC_{n+1}(X)) \\
M(PC_n(X)) \arrow[u, "\text{choose } * \in N_0(i)"] \arrow[ur, bend right] &
\end{tikzcd}$$

$\square$.

Coro. This induces isoms on H^{pq}_{et} for $p \leq n/2$.

Proof Follows from

- $\exists$ of motivic stab maps
- Prop ① from "Detecting isomorphisms"
- the classical result of McDuff-Segal for $X_{\mathbb{C}}$.

Pure motives & Postnikov filtration.

Def (rough idea):

$M \in \mathrm{Mot}$ is p/q-pure ($p/q \in \mathbb{Q}_{>0}$)

if

$$
\begin{array}{ccc}
W_{\geq 3}(M) & & \\
\downarrow & & \\
\vdots & & \\
\downarrow & \overset{B}{\longmapsto} & \\
W_{\geq 1}(M) & & \\
\downarrow & & \\
W_{\geq 0}(M) & & \\
\downarrow & & \downarrow \\
M & & B(M)
\end{array}
$$

Postnikov filtration of $B(M)$, rescaled by p/q.

$\boxed{\mathrm{Mot}}$ $\qquad$ $\boxed{D(\mathrm{Chain})}$

<u>Lem</u> If M is p/q - pure, then

$$B(w_s(M)) = 0 \qquad \text{if} \qquad s \neq qm$$

$$B(w_{qm}(M)) \cong H_{pm}(B(M)) \quad \text{concentrated in degree } pm.$$

<u>Eg</u> $M = M(\mathbb{P}^m)$ is 2-pure

<u>Prop</u> If X = complement of codim-d hyperplanes in $\mathbb{A}^m$, then $M(X)$ is $\left(\frac{2d-1}{d}\right)$-pure.

(pf — induction on # of hyperplanes + formal properties of purity).

THS for Σ_n

Thm (P. '18) if (T_n) is a sequence of Σ_n-modules forming a polynomial coeff system of degree $\leq k$

then $H_q(\Sigma_n; T_n) \longrightarrow H_q(\Sigma_{n+1}; T_{n+1})$

is an isom. for $q \leq \frac{n-k}{2}$.

Prop: Fix $m \geq 2$ and $r \geq 0$.

Then $H_r(\mathrm{PConf}_n(\mathbb{R}^m))$ forms a poly coeff system of degree $\leq \frac{2r}{m-1}$.

proof: (some work) $\Rightarrow$ sufficient to show that

$$\operatorname{rank}_{\mathbb{Z}}\left(H_r(\mathrm{PConf}_n(\mathbb{R}^m))\right) \sim O\left(n^{\frac{2r}{m-1}}\right).$$

Thm [F. Cohen '76] $H^*\mathrm{PConf}_n(\mathbb{R}^m)$ is generated as a ring by $\binom{n}{2}$ elements in degree $m-1$.

Hence $\operatorname{rank}_{\mathbb{Z}}\left(H^{i(m-1)}(\mathrm{PConf}_n(\mathbb{R}^m))\right) \leq$ # of degree-i monomials in $\binom{n}{2}$ variables

$$\leq \binom{n}{2}^i \sim O(n^{2i}).$$

$\square$.

Motivic stability for $C_n(\mathbb{A}^d)$

By Prop ②, sufficient to prove stability ($p \leq n/2$) for

$$H^p(B(w_s(\underline{M(C_n(\mathbb{A}^d))}))) \qquad \forall s \geq 0$$

B and w_s commute with htpy quotients by Σ_n

$$H^p(\, B(w_s(M(PC_n(\mathbb{A}^d))))_{\Sigma_n}\,)$$

$$H_p(\text{———}) = (*)$$

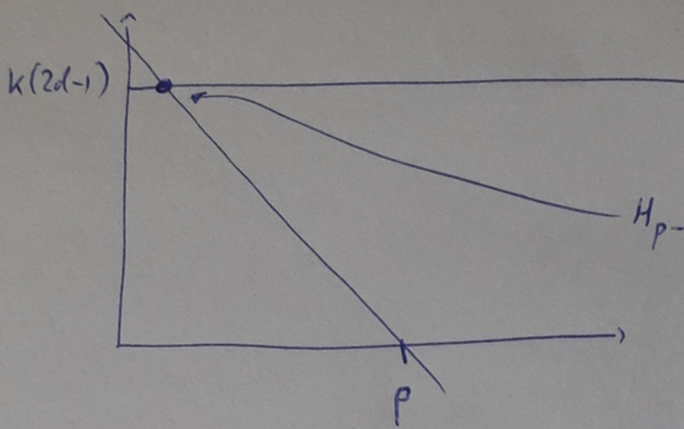
$$H_*(\text{———}) \Leftarrow H_*(\Sigma_n ; H_*(Bw_s MPC_n(\mathbb{A}^d)))$$

$PC_n(\mathbb{A}^d)$ is a codim-d hyperplane complement

so $MPC_n(\mathbb{A}^d)$ is $\frac{2d-1}{d}$-pure

$$\text{so}\quad Bw_s MPC_n(\mathbb{A}^d) = \begin{cases} 0 & s \neq kd \\ H_{k(2d-1)}(PC_n(\mathbb{C}^d)) & s = kd \end{cases}$$

concentrated in degree $k(2d-1)$.



$H_{p-k(2d-1)}(\Sigma_n; \blacksquare)$

(s=kd)

$$H_{k(2d-1)}(PC_n(\sigma^d))$$
$$\shortparallel$$
$$H_{k(2d-1)}(PC_n(\mathbb{R}^{2d})).$$

So $(*) = H_{p-k(2d-1)}\left(\Sigma_n;\, H_{k(2d-1)}(PC_n(\mathbb{R}^{2d}))\right)$

$$\frac{2r}{m-1} = \frac{2k(2d-1)}{2d-1} = 2k$$

THS $\rightsquigarrow$ stab^y for

$$p - k(2d-1) \leq \frac{n - \text{degree}}{2}$$

$$\Updownarrow$$

$$p \leq \frac{n}{2} + \underbrace{k(2d-2)}_{\geq\, 0}.$$

$\square$.