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Kontsevich characteristic classes à la Watanabe

reading seminar
Oxford

12 May 2020

Main reference: [W, §2]

[W] = "Some exotic nontrivial elements of the rational
homotopy groups of $\text{Diff}(S^4)$ "

Tadayuki Watanabe

arXiv: 1812.02448

v1: December 2018

v3: August 2019

(most recent at time of talk!)

Kontsevich characteristic classes à la Watanabe

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①

ref. [W, §2.1]

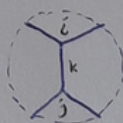
Trivalent graphs

Definition $A_k := \mathbb{Q} \left\{ \begin{array}{l} \text{connected trivalent graphs with } 2k \text{ vertices (hence } 3k \text{ edges)} \\ \text{with } \boxed{\text{vertices labelled by } \{1, \dots, 2k\}} \\ \text{edges labelled by } \{1, \dots, 3k\}} \end{array} \right\} \sim$

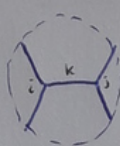
where $\sim$ is generated by

(IHX)

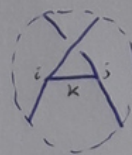
(i)



+



+



~ 0

(ii)

$\Gamma + \Gamma'$

~ 0

exchange labels of two edges

(iii)

$\Gamma - \Gamma''$

~ 0

exchange labels of two vertices

Fact :

We have :

k	1	2	3	4	5	6	7	8	9
$\dim_{\mathbb{Q}}(A_k)$	0	1	0	0	1	0	0	0	1



P.T.O.

[Bar-Natan, McKay]

$$\frac{2}{3} \left(\text{Diagram 1} \right) + \left(\text{Diagram 2} \right) + \frac{4}{3} \left(\text{Diagram 3} \right) + 2 \left(\text{Diagram 4} \right) + \left(\text{Diagram 5} \right)$$

Diagram 1: A circle containing a central cube-like structure. The regions are numbered 1 through 15. The regions are arranged in a way that suggests a 3D object, with the central cube having faces numbered 1 through 6, and the surrounding regions numbered 7 through 15.

Diagram 2: A hexagon with a vertical line through the center. The regions are numbered 1 through 15. The regions are arranged in a way that suggests a 3D object, with the central cube having faces numbered 1 through 6, and the surrounding regions numbered 7 through 15.

Diagram 3: A hexagon with a central vertical line. The regions are numbered 1 through 15. The regions are arranged in a way that suggests a 3D object, with the central cube having faces numbered 1 through 6, and the surrounding regions numbered 7 through 15.

Diagram 4: A hexagon with a central vertical line. The regions are numbered 1 through 15. The regions are arranged in a way that suggests a 3D object, with the central cube having faces numbered 1 through 6, and the surrounding regions numbered 7 through 15.

Diagram 5: A complex shape with a central square. The regions are numbered 1 through 15. The regions are arranged in a way that suggests a 3D object, with the central cube having faces numbered 1 through 6, and the surrounding regions numbered 7 through 15.

Larger context:

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Graph complex $\mathcal{G} = \bigoplus_{d,e \geq 0} \mathcal{G}_d^e$

$$\mathcal{G}_d^e := \mathbb{Q} \left\{ \text{all graphs with } \begin{array}{l} \#E - \#V = d \\ 2\#E - 3\#V = e \end{array} \right\}$$

no multiple edges
no loops
all vertices of valence ≥ 3

degree

excess

(note: trivalent $\Leftrightarrow$ excess = 0)

This has a differential $\mathcal{G}_d^e \rightarrow \mathcal{G}_d^{e-1}$ given by edge contraction.

Remark: $H_k^0(\mathcal{G}) = \ker(\mathcal{G}_k^0 \rightarrow 0) / \text{im}(\mathcal{G}_k^1 \rightarrow \mathcal{G}_k^0)$

$\uparrow$
(IHX)

$$\cong A_k$$

Kontsevich char. classes more generally take values in $H_*^*(\mathcal{G})$,

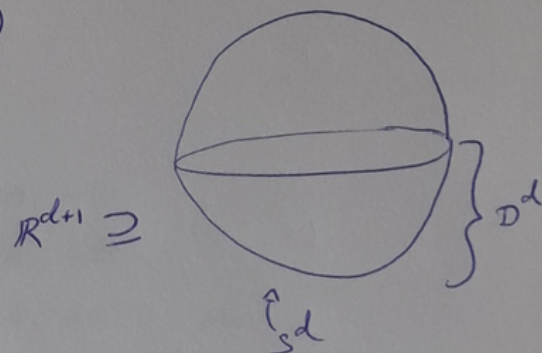
but we (and Watanabe) just need $H_*^0(\mathcal{G}) \cong A_*$.

Aim: Understand $\pi_* \text{Diff}(S^4)$

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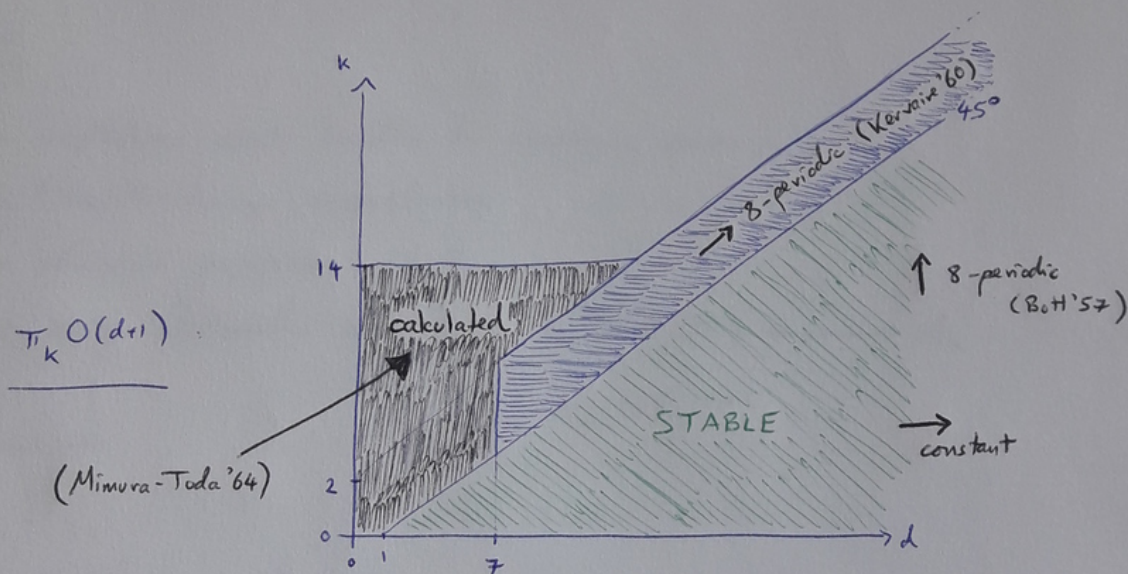
In general:

$$\begin{array}{ccc} \text{Diff}_0(D^d) & \xleftarrow{\text{extend by id.}} & \text{Diff}(S^d) \\ & \nearrow \text{restrict} & \\ O(d+1) & & \end{array}$$



Fact:

$$\text{Diff}_0(D^d) \times O(d+1) \xrightarrow{\cong} \text{Diff}(S^d)$$



$$\underline{d \leq 3} \quad \text{Diff}_0(D^d) \cong *$$

(Smale '59, Hatcher '83)
d=2 d=3

$$\underline{\pi_k \text{Diff}_0(D^d):}$$

$$\underline{d \geq 5} \quad \pi_0 \text{Diff}_0(D^d) \cong \mathbb{H}_{d+1} = \frac{\{\text{homotopy } (d+1)\text{-spheres}\}}{\text{diffeo}^+}$$

(Smale '62 + Cerf '70)
(surj) (inj)

- finite [Kervaire-Milnor '63]
- $\neq 0$ for all $d \in \{6, 7, \dots, 61\}$ except 11, 55 and 60 and all even $d \geq 62$.

d=4 (Watanabe '18):

$$\begin{array}{ccc} \pi_{k-1} \text{Diff}_2(D^4) & \xrightarrow{\hat{Z}_k} & A_k \\ \downarrow & \nearrow & \text{surjective} \\ \pi_{k-1} \text{Diff}_2(D^4) \otimes \mathbb{Q} & & \end{array}$$

Corollary: $\pi_1 \text{Diff}_2(D^4) \neq 0$ since $A_2 \neq 0$
 $\pi_4 \text{Diff}_2(D^4) \neq 0$ since $A_5 \neq 0$
 $\pi_8 \text{Diff}_2(D^4) \neq 0$ since $A_9 \neq 0$ etc.

Outline

- recollections about bundles & classifying spaces
- Fulton-MacPherson compactifications
- admissible propagators and Z_k
- iteration of bundles and $\hat{Z}_k = \pi_k \text{BDiff}_2(D^4) \longrightarrow A_k$
 $\parallel$
 $\pi_{k-1} \text{Diff}_2(D^4)$

next week
 $\downarrow$

- well-defined-ness of $\hat{Z}_k$
- why it factors through $\text{image} \left(\pi_k \text{BDiff}_2(D^4) \xrightarrow{H} \Omega_k(\text{BDiff}_2(D^4)) \right)$
- more formulas for $\hat{Z}_k$ on $\text{image}(H)$...

oriented bordism group

② Bundles and associated principal bundles

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ref: Steenrod
"The topology of fibre bundles"
(1951)

Def: $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ fibre bundle with fibre F
& structure group $G \curvearrowright F$

• locally looks like $\begin{array}{c} F \times U \\ \downarrow \\ U \end{array}$ for $U \subseteq X$

• transition functions act on F through G .

Eg: $GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$ vector bundles
 $Diff(M) \curvearrowright M$ smooth M -bundles
 $F = G \curvearrowright G$ principal G -bundles

Lem: If $\begin{array}{c} E \\ \downarrow \\ X \end{array}$ is a principal G -bundle, there is an action of G on E ,
and $X = E/G$

moreover:

$$\begin{array}{ccc} \{\text{principal } G\text{-bundles } / X\} / \cong & \xrightarrow[\text{(*)}]{\text{Borel construction}} & \{\text{fibre bundles } / X \text{ with fibre } F \text{ and str. group } G\} / \cong \\ \begin{array}{c} E \\ \downarrow \\ X \end{array} & \xrightarrow{\quad} & \begin{array}{c} E \times_G F = (E \times F) / G \\ \downarrow \\ X \end{array} \end{array}$$

Prop: The Borel construction is a bijection.

Def: The pre-image of a fibre bundle under (*) is its associated principal bundle.

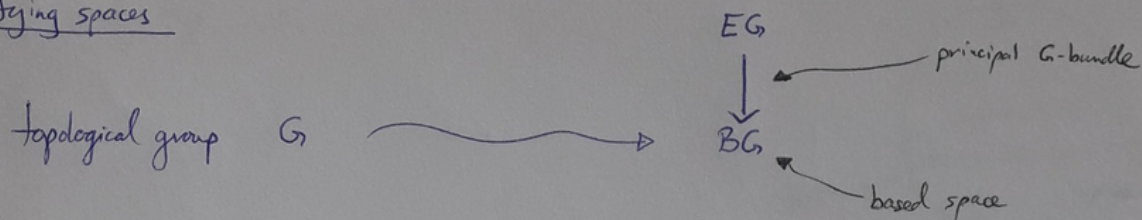
③ Bundles and classifying spaces

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ref: Milnor

"Construction of universal bundles II"
(1956)

Classifying spaces



Thm: $[X, BG] \xrightarrow{\cong} \{ \text{principal } G\text{-bundles over } X \} / \cong$ (X paracompact)

$$[f: X \rightarrow BG] \mapsto \begin{array}{c} f^*(EG) \\ \downarrow \\ X \end{array}$$

Lemma: $\Omega(BG) \cong G$

hence $\pi_{k-1}(G) \cong \pi_k(BG) \quad \forall k$

Coro: M, X manifolds

$$[(X, \partial X), (B\text{Diff}_0(M), *)] \xrightarrow{\cong} \left\{ \begin{array}{c} \text{smooth } M\text{-bundles} \\ \downarrow \\ X \\ \text{with } \partial E = \partial(M \times X) \end{array} \right\} / \cong$$

$\partial^{\text{fib}} E = \partial^h E$

Special case: $X = [0, 1]^k = I^k$
 $M = D^d$

$$\begin{array}{c} \pi_k B\text{Diff}_0(D^d) \\ \cong \\ \pi_{k-1} \text{Diff}_0(D^d) \end{array} \cong \left\{ \begin{array}{c} \text{smooth } D^d\text{-bundles} \\ \downarrow \\ I^k \\ \text{with } \partial E = \partial(D^d \times I^k) \end{array} \right\} / \cong$$

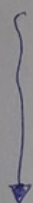
④ Fulton - MacPherson compactifications and the map $\partial \bar{C}_2(S^d, \infty) \rightarrow S^{d-1}$

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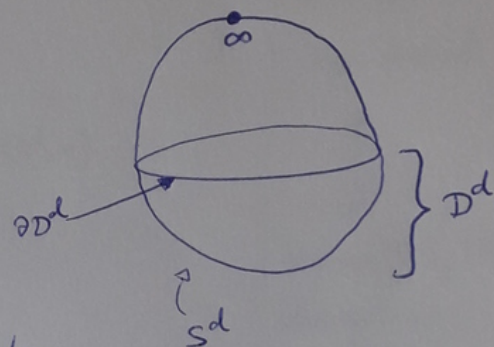
ref: [W, §2.2
§2.3]

Def $C_n(S^d, \infty) = \{ \text{ordered configurations of } n \text{ distinct points in } S^d - \{\infty\} \}$

$\text{Diff}_0(D^d)$



$\bar{C}_n(S^d, \infty) = \text{its Fulton - MacPherson compactification}$



Key properties

- smooth compact manifold with corners

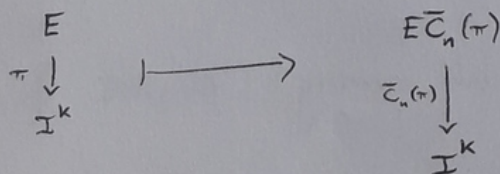
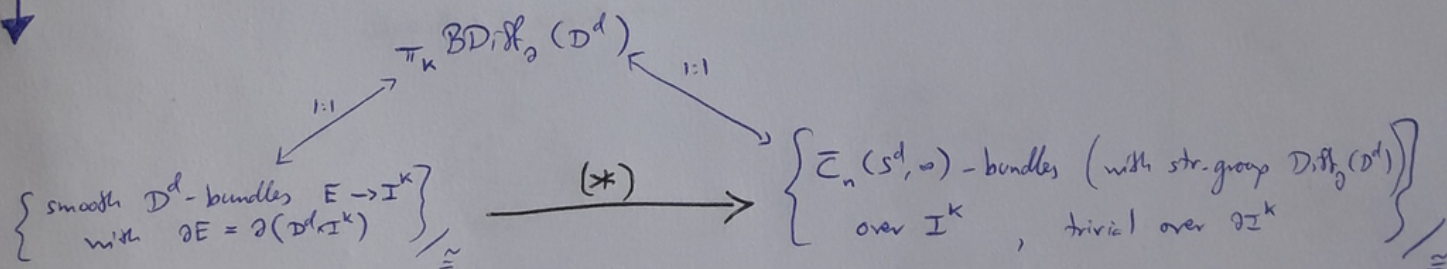
↳ boundary strata $\longleftrightarrow$ subsets $A \subseteq \{1, 2, \dots, n, \infty\}$ $|A| \geq 2$
 $\emptyset \neq$

- its interior is $C_n(S^d, \infty)$

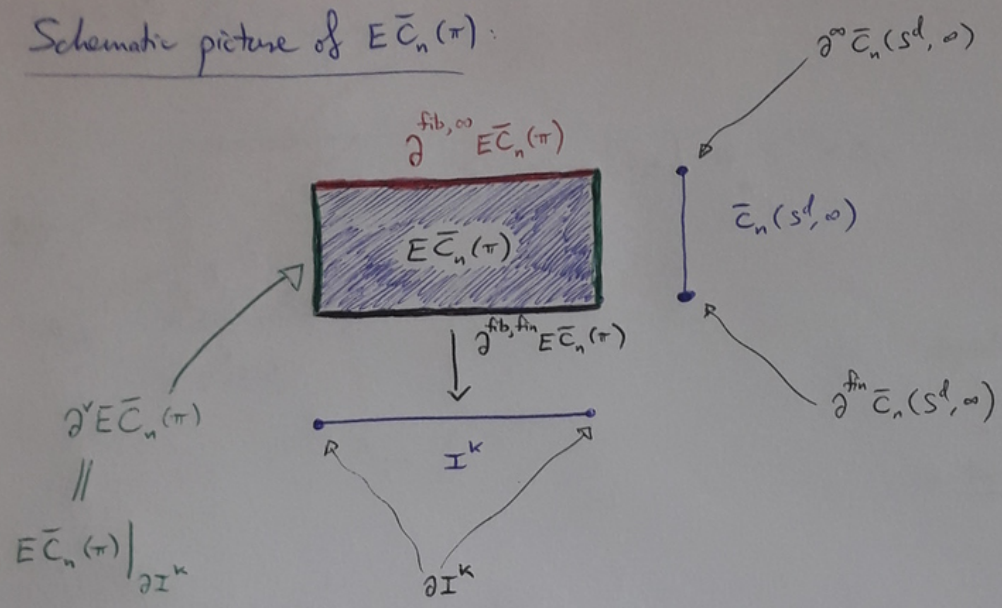
- the inclusion $C_n(S^d, \infty) \hookrightarrow \bar{C}_n(S^d, \infty)$ is a homotopy equivalence

- the action of $\text{Diff}_0(D^d)$ extends naturally to $\bar{C}_n(S^d, \infty)$

- the action of Σ_n extends naturally to $\bar{C}_n(S^d, \infty)$



Schematic picture of $E\bar{C}_n(\pi)$:



- is trivial
 - & are non-trivial
 because $Diff_0(D^d)$ does not act trivially on $\partial \bar{C}_n(s^d, \infty)$
 (only on the stratum corresponding to $A = \{1, 2, \dots, n, \infty\}$)

where $\partial^{fib} E\bar{C}_n(\pi) = \partial^{fib, \infty} E\bar{C}_n(\pi) \cup \partial^{fib, fin} E\bar{C}_n(\pi)$

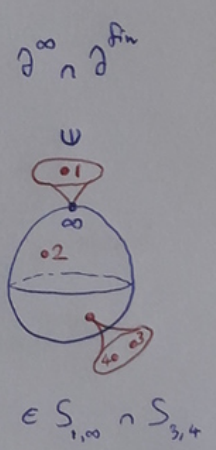
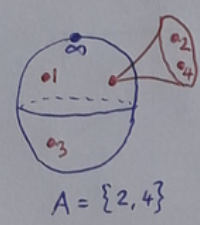
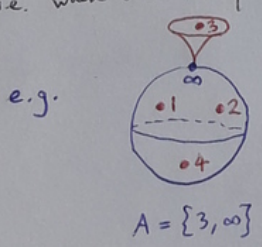
Explanations:

$$\partial \bar{C}_n(s^d, \infty) = \partial^{\infty} \bar{C}_n(s^d, \infty) \cup \partial^{fin} \bar{C}_n(s^d, \infty)$$

all strata corresponding to subsets $A \subseteq \{1, 2, \dots, n, \infty\}$ with $\infty \in A$

all other strata (no point at ∞ ; just "collisions")

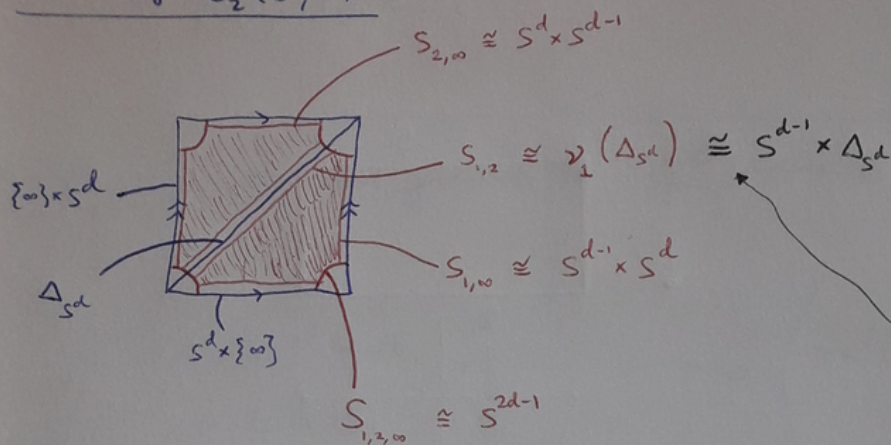
i.e. where there is a point at ∞



The action of $Diff_0(D^d)$ preserves this decomposition.

Picture of $\bar{C}_2(S^d, \infty)$:

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depends on choice of
framing of $S^d - \{\infty\}$
= trivialisation of $T(S^d - \{\infty\})$

The map $\partial \bar{C}_2(S^d, \infty) \rightarrow S^{d-1}$:

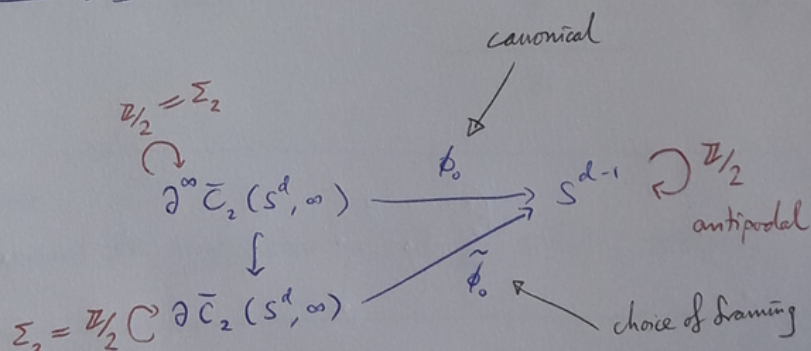
$$\begin{aligned} \bullet \quad S_{1,2,\infty} - \Delta_{S^d} &\cong S^{2d-1} - \Delta_{S^d} \\ &\subseteq (\mathbb{R}^d)^2 - \Delta_{\mathbb{R}^d} \xrightarrow{(x,y) \mapsto \frac{x-y}{|x-y|}} S^{d-1} \end{aligned}$$

$$\bullet \quad S_{1,\infty} \cong S^{d-1} \times S^d \xrightarrow{\text{proj}_1} S^{d-1}$$

$$\bullet \quad S_{2,\infty} \cong S^d \times S^{d-1} \xrightarrow{\text{proj}_2} S^{d-1}$$

$$\bullet \quad S_{1,2} \cong \underbrace{S^{d-1} \times \Delta_{S^d}}_{\text{choice of framing}} \xrightarrow{\text{proj}_1} S^{d-1}$$

Lemma : (1) These glue together to



(2) ϕ_0 and $\tilde{\phi}_0$ are $\mathbb{Z}_2$ -equivariant

(3) ϕ_0 extends canonically to $\partial^{\text{fib}, \infty} E \bar{C}_2(\pi) \xrightarrow{\phi_0} S^{d-1}$

(4) given a choice of trivialisation of $\begin{matrix} T^*E \\ \downarrow \\ E \end{matrix}$ (vertical tangent bundle of $\begin{matrix} E \\ \downarrow \\ I^k \end{matrix}$),

$\tilde{\phi}_0$ extends to $\partial^{\text{fib}} E \bar{C}_2(\pi) \xrightarrow{\tilde{\phi}_0} S^{d-1}$

vertical framing

⑤ Admissible propagators

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ref: [W, §2-3]

Note: Everything above generalises to $I^k \rightsquigarrow B$ any k -manifold with corners.

$$\left(\begin{array}{c} \text{smooth } D^d\text{-bundle} \\ \text{with } \partial E \text{ trivial} \end{array} \begin{array}{c} E \\ \pi \downarrow \\ B \end{array} \right) \longrightarrow \left(\begin{array}{c} \bar{C}_n(S^d, \infty)\text{-bundle} \\ \bar{C}_n(\pi) \downarrow \\ B \end{array} \begin{array}{c} E\bar{C}_n(\pi) \\ \bar{C}_n(\pi) \downarrow \\ B \end{array} \right)$$

All chain complexes over $\mathbb{Q}$.

Def: An admissible propagator for a smooth D^d -bundle $\begin{array}{c} E \\ \pi \downarrow \\ B^k \end{array}$ is:

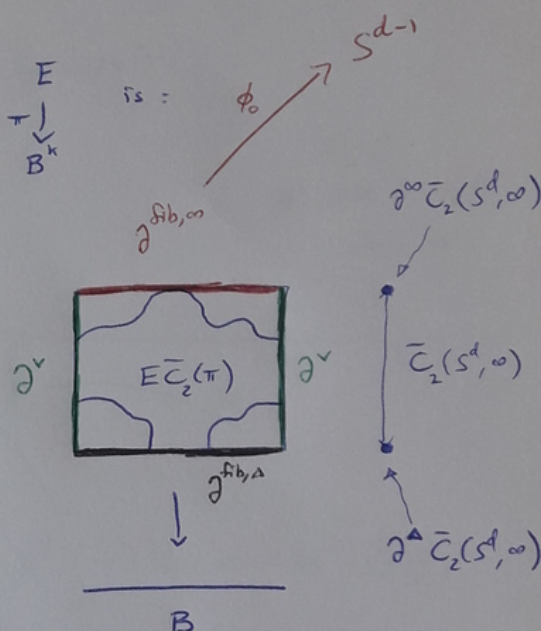
$$\theta \in Z_{k+d+1}(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi))$$

satisfying

(i) θ transverse to $\partial^\vee E\bar{C}_2(\pi)$

(ii) $\partial\theta|_{\bullet}$ is $\mathbb{Z}_2$ -invariant

(iii) $\partial\theta|_{\bullet} = \phi_0^{-1}(\{\pm a\})$, some $a \in S^{d-1}$



Proposition: (a) such a θ exists
(b) all such θ represent the same class in $H_{k+d+1}(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi))$

Rmk: If $\begin{array}{c} E \\ \pi \downarrow \\ X \end{array}$ is equipped with a vertical framing, then

- $\tilde{\phi}_0: \partial^{\text{fib}} E\bar{C}_2(\pi) \longrightarrow S^{d-1}$ is defined

- strong admissible propagator: replace (ii) + (iii) with (iv) $\partial\theta|_{\bullet} = \tilde{\phi}_0^{-1}(\{\pm a\})$

- there also exist & are unique upto homology.

Proof: (For $B = \mathbb{I}^k$, to simplify the argument)

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- Enough to show that $\exists$ unique-up-to-homology θ satisfying (iii)

$$\left(\begin{array}{l} \text{Then transversality} \longrightarrow (i) \\ c \mapsto \frac{1}{2}(c + \tau.c) \longrightarrow (ii) \end{array} \right)$$

- Re-interpret (iii) in terms of (co)homology:

$$\begin{array}{ccccc} & & [\partial\theta] & \xleftarrow{\quad} & [\theta] \\ & \nearrow & \cap & & \cap \\ & & H_{k+d}(\partial E \bar{C}_2(\pi)) & \xleftarrow{\delta} & H_{k+d+1}(E \bar{C}_2(\pi), \partial E \bar{C}_2(\pi)) \\ & \searrow & \cong & & \cong \\ \phi_0^*(1) & & H^{d-1}(\partial E \bar{C}_2(\pi)) & \xleftarrow{(incl)^*} & H^{d-1}(E \bar{C}_2(\pi)) \\ \cap & & \cap & & \cap \\ H^{d-1}(\partial^{db, \infty} E \bar{C}_2(\pi)) & \xleftarrow{(incl)^*} & H^{d-1}(\partial E \bar{C}_2(\pi)) & \xleftarrow{(incl)^*} & H^{d-1}(E \bar{C}_2(\pi)) \\ \uparrow \phi_0^* & & \nwarrow & & \nwarrow \\ 1 \in H^{d-1}(S^{d-1}) = \mathbb{Q} & & & & \end{array}$$

(*)

Need:
 $\exists!$ lift of $\phi_0^*(1)$
along (*)

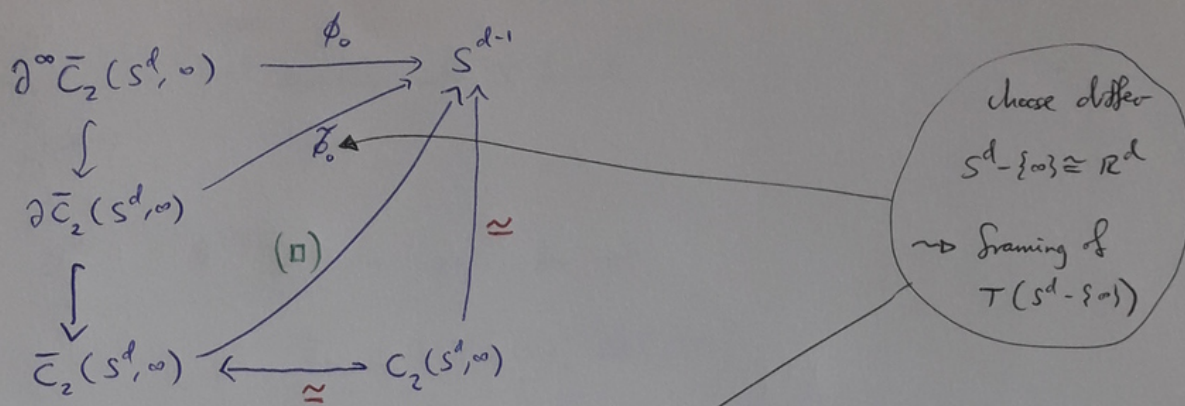
$$\begin{array}{ccc} S^{d-1} & \xleftarrow{(\square)} & \partial^\infty \bar{C}_2(S^{d, \infty}) \longleftrightarrow \bar{C}_2(S^{d, \infty}) \\ \uparrow \phi_0 & \searrow \phi_0 & \downarrow \cong \\ \partial^{db, \infty} E \bar{C}_2(\pi) & \longleftrightarrow & E \bar{C}_2(\pi) \\ & \searrow & \downarrow \\ & & B = \mathbb{I}^k \end{array}$$

$$\xrightarrow{H^{d-1}} \begin{array}{ccc} 1 \in H^{d-1}(S^{d-1}) & \xrightarrow{\cong} & H^{d-1}(-) \xleftarrow{\quad} H^{d-1}(-) \\ & \searrow & \uparrow \cong \\ & \phi_0^*(1) \in H^{d-1}(-) & \xleftarrow{(*)} H^{d-1}(-) \end{array}$$

Claim: $\exists$ map $(\square)$, which is a homotopy equivalence

proof of claim:

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($\square$) defined on the interior by

$$C_2(S^d, \infty) \cong C_2(\mathbb{R}^d) \xrightarrow{\cong} S^{d-1}$$

$$(x, y) \longmapsto \frac{x-y}{|x-y|}$$

$\square$.

⑥ Z_k

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Fix $\begin{matrix} E \\ \pi \downarrow \\ B^{k(d-3)} \end{matrix}$ smooth D^d -bundle with ∂E trivial

Choose $\theta^{(1)} \dots \theta^{(3k)}$ adm. prog. for π

$$\in Z_{k(d-3)+d+1}(\bar{E\bar{C}}_2(\pi), \partial \bar{E\bar{C}}_2(\pi))$$

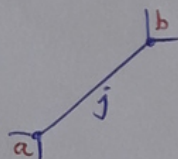
codim = $d-1$

Γ trivalent graph with vertices $\longleftrightarrow \{1 \dots 2k\}$
edges $\longleftrightarrow \{1 \dots 3k\}$

$1 \leq j \leq 3k$

$$\phi_j: E\bar{C}_{2k}(\pi) \longrightarrow E\bar{C}_2(\pi) \quad \text{with } \mathbb{Z}/2$$

forget all points except those labelled by a and b



$$\longrightarrow \phi_j^{-1}(\theta^{(j)}) \quad \text{codimension } d-1 \text{ chain on } E\bar{C}_{2k}(\pi)$$

$\mathbb{Z}/2$ -invariant

$\uparrow$ in Γ

Assume that $\phi_1^{-1}(\theta^{(1)}) \dots \phi_{3k}^{-1}(\theta^{(3k)})$ are mutually transverse

$$\longrightarrow \mathbb{I}(\Gamma) := \bigcap_{j=1}^{3k} \phi_j^{-1}(\theta^{(j)}) \quad \text{codim } 3k(d-1) \text{ chain on } E\bar{C}_{2k}(\pi)$$

$\in \mathbb{Q}$

$$\longrightarrow Z_k := \lambda_k \sum_{\Gamma} \mathbb{I}(\Gamma) \cdot [\Gamma] \in A_k$$

$$\begin{aligned} \dim &= k(d-3) + 2kd \\ &= 3k(d-1) \end{aligned}$$