Kontsevich characteristic classes à la Watanabe

Part 2

reading seminar
Oxford
19 May 2020

Main reference: §2 of "Some exotic nontrivial elements of the rational homotopy groups of $\mathrm{Diff}(S^4)$"

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arXiv: 1812.02448 (currently v3, from August 2019)

Plan:

- brief recollections from last week
- microscopic configuration spaces on $\mathbb{R}^d$
- constructions
  - Z_k (D^4- bundle over $B^{k+\ell}$) [with trivialized ∂]
  - α_k ($\mathbb{R}^4$-bundle over $B^{k+s+\ell}$)
  - $\hat{Z}_k$ (D^4- bundle over $[0,1]^k$) [with trivialized ∂]

- Thm: $\hat{Z}_k$ well-defined function $\pi_k BDiff_\partial(D^4) \longrightarrow A_k$
- Lemma: $\hat{Z}_k$ is a homomorphism
- proof
- Prop: $\hat{Z}_k$ factors through image $\left(\pi_k BDiff_\partial(D^4) \xrightarrow{H} \Omega_k(BDiff_\partial(D^4)) \right)$

oriented bordism classes of smooth D^4-bundles over S^k

Outlook:

$$\begin{tikzcd}
\pi_k BDiff_\partial(D^4) \arrow[rr, "\hat{Z}_k \ \boxed{\S 2}"] \arrow[dr] & & A_k = H^0_k(\mathcal{G}) \\
& \pi_k BDiff_\partial(D^4) \otimes \mathbb{Q} \arrow[ur] & \mathcal{G}^0_k \arrow[u, two heads] \\
& & \bar{\mathcal{G}}^0_k \arrow[u, hook] \arrow[ul, bend left, "\boxed{\S 4}"] \arrow[uu, bend right]
\end{tikzcd}$$

$\boxed{\S 5}$

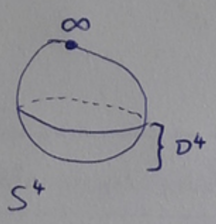
$\boxed{\S 3}$ graph-counting formula

Recollections

$$\mathrm{Graphs}^{\circ}_{k} = \left\{ \text{trivalent graphs with } \begin{matrix} \text{vertices} \leftrightarrow \{1 \ldots 2k\} \\ \text{edges} \leftrightarrow \{1 \ldots 3k\} \end{matrix} \right\}$$

$$= \left\{ \text{2-valued functions } \{1 \ldots 3k\} \dashrightarrow \{1 \ldots 2k\} \right\}$$

$$A_k = \mathbb{Q}\langle \mathrm{Graphs}^{\circ}_{k} \rangle \Big/ \left(\begin{matrix} \text{IHX relation}, \ \Gamma \sim \Gamma \text{ with 2 vertices swapped}, \\ -\Gamma \sim \Gamma \text{ with 2 edges swapped} \end{matrix} \right)$$



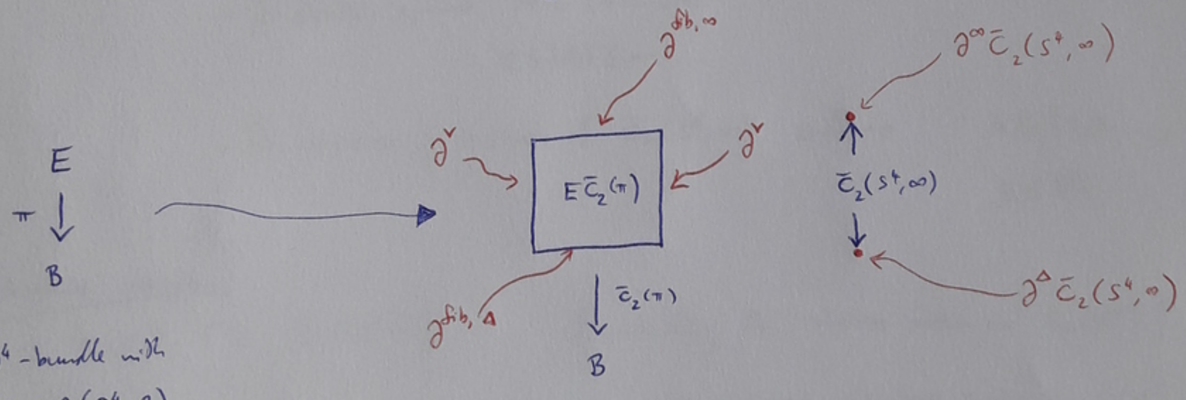
$\overline{C}_n(S^4, \infty)$ = Fulton MacPherson compactification of $C_n(S^4 - \{\infty\})$

↖ ordered configuration space.

Facts:

- $$\underbrace{\partial \overline{C}_2(S^4,\infty)}_{\substack{\cong \ *_3 \\ S^3 \times S^4}} = \underbrace{\partial^{\infty} \overline{C}_2(S^4,\infty)}_{\substack{\cong \ *_1 \\ S^3 \times D^4}} \cup \underbrace{\partial^{\Delta} \overline{C}_2(S^4,\infty)}_{\substack{\cong \ *_2 \\ S^3 \times D^4}}$$

$$S^3 \times S^4 \xrightarrow{\phi_0 = \mathrm{proj}_1} S^3$$



smooth D^4-bundle with $\partial E = \partial(D^4 \times B)$

- the maps $\partial^{\infty}\overline{C}_2(S^4,\infty) \xrightarrow{\phi_0} S^3$ on each fibre fit together to give a continuous map $\partial^{fib,\infty} E\overline{C}_2(\pi) \xrightarrow{\phi_0} S^3$ (the identifications $*_1$ can be made <u>consistently</u>)
- <u>if</u> we choose a <u>vertical framing</u> of $\begin{smallmatrix} E \\ \downarrow \pi \\ B \end{smallmatrix}$, <u>then</u> the identifications $*_2$ (and hence $*_3$) can also be made canonically for each fibre, so the maps ϕ_0 fit together to give a continuous map $\partial^{fib} E\overline{C}_2(\pi) \xrightarrow{\phi_0} S^3$.

An <u>admissible propagator</u> for $\begin{array}{c} E \\ \pi\downarrow \\ B \end{array}$ is a codim-3 chain θ on $E\bar{C}_2(\pi)$

with $\partial\theta \subseteq \partial E\bar{C}_2(\pi)$

$\left[\text{so } \theta \in Z_{k+5}\left(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi)\right)\right]$

such that

- θ is transverse to ∂^v
- $\partial\theta\big|_{\partial^{fib,\infty}} = \phi_0^{-1}(\{\pm a\}) \qquad a \in S^3$
- $\partial\theta\big|_{\partial^{fib,\Delta}}$ is $\mathbb{Z}/2$-invariant

A <u>strong admissible propagator</u>

depends on choice of vertical framing of $\begin{array}{c} E \\ \pi\downarrow \\ B \end{array}$

- $\partial\theta\big|_{\partial^{fib}} = \phi_0^{-1}(\{\pm a\})$

Microscopic configuration spaces on $\mathbb{R}^d$

<u>Def</u>: $$S_n(\mathbb{R}^d) = \left\{ (x_1, \dots, x_n) \in C_n(\mathbb{R}^d) \;\middle|\; \begin{array}{c} x_1 = 0 \\ \sum_i |x_i|^2 = 1 \end{array} \right\}$$

dimension $= d(n-1) - 1$

<u>Compactification</u>: $\bar{S}_n(\mathbb{R}^d) :=$ closure of $S_n(\mathbb{R}^d) \subseteq C_n(\mathbb{R}^d) \subseteq \bar{C}_n(S^d, \infty)$

- compact manifold with corners
- ∂-strata $\xleftrightarrow{1:1}$ $A \subseteq \{1, 2, \dots, n\}$, $\quad 2 \leq |A| \leq n-1$
- for comparison, ∂-strata of $\bar{C}_n(S^d, \infty)$ $\xleftrightarrow{1:1}$ $A \subseteq \{1, 2, \dots, n, \infty\}$, $\quad 2 \leq |A|$

<u>Fibrewise version</u>:

- $O(d) \curvearrowright \bar{S}_n(\mathbb{R}^d)$ $\quad$ [extending the obvious action on $S_n(\mathbb{R}^d)$]
- $\begin{array}{c} V \\ \xi\downarrow \\ B \end{array}$ rank-d real vector bundle $\longrightarrow$
  - wlog assume structure group is $O(d)$
  - take associated principal $O(d)$-bundle
  - Borel construction $\rightsquigarrow$ change fibre to $\bar{S}_n(\mathbb{R}^d)$

$$\begin{array}{c} \bar{S}_n(V) \\ \big\downarrow \bar{S}_n(\xi) \\ B \end{array}$$

<u>intuitively</u>: $\bar{S}_n(V) = \bigcup_{b \in B} \bar{S}_n(\xi^{-1}(b))$

From now on, $d=4$

Construction of Z_k:

Inputs

$$\boxed{\begin{array}{ll} E & \text{smooth } D^4\text{-bundle} \\ \downarrow \pi & \text{with } \partial E = \partial(D^4 \times B) \\ B^{k+\ell} & \end{array}} \quad \xleftrightarrow{1:1} \quad [(B,\partial B), (B\mathrm{Diff}_\partial(D^4), *)]$$

$$\boxed{\theta = (\theta^{(1)}, \dots, \theta^{(3k)}) \text{ admissible propagators for } \pi}$$

For $A \subseteq \{1, \dots, 2k\}$, $|A| = 2$ we have forgetful maps

$$\begin{tikzcd} E\bar{C}_{2k}(\pi) \arrow[rr, "\phi_A"] \arrow[dr] & & E\bar{C}_2(\pi) \arrow[dl] \\ & B & \end{tikzcd}$$

For a trivalent graph $\Gamma \in \mathrm{Graphs}_k^\circ$, define

$$\boxed{I(\pi, \theta, \Gamma) := \bigcap_{j=1}^{3k} \phi_{\Gamma(j)}^{-1}(\theta^{(j)}) \quad \in \quad C_\ell(E\bar{C}_{2k}(\pi))}$$

(assuming that the intersection is transverse)

Then define

$$\boxed{Z_k(\pi, \theta) := \lambda_k \sum_{\Gamma \in \mathrm{Graphs}_k^\circ} I(\pi, \theta, \Gamma) \otimes [\Gamma] \quad \in \quad C_\ell(E\bar{C}_{2k}(\pi)) \underset{\mathbb{Q}}{\otimes} \mathcal{A}_k}$$

$$\boxed{\lambda_k = \frac{1}{2^{3k}(2k)!(3k)!}}$$

When $\ell = 0$ (i.e. $\dim(B) = k$):

$$\begin{array}{ccc} Z_k(\pi,\theta) & & \\ \cap & & \\ C_0(E\bar{C}_{2k}(\pi)) \underset{\mathbb{Q}}{\otimes} \mathcal{A}_k & \xrightarrow{\mathrm{aug} \otimes \mathrm{id}} & \mathbb{Q} \underset{\mathbb{Q}}{\otimes} \mathcal{A}_k = \mathcal{A}_k \end{array}$$

By abuse of notation, we denote the image also by $\boxed{Z_k(\pi, \theta) \in \mathcal{A}_k}$.

Construction of α_k:

Inputs

$$\boxed{\begin{array}{ll} V & \text{rank-4 real} \\ \downarrow \xi & \text{vector bundle} \\ X^{k+5+\ell} & \end{array}} \quad \overset{1:1}{\longleftrightarrow} \quad [X, BO(4)]$$

NB: Reducing the structure group $GL_d(\mathbb{R}) \rightsquigarrow O(d)$ corresponds to choosing a Riem. metric for the vector bundle.
→ The space of Riem. metrics is contractible
→ After choosing this, we may speak of "unit sphere sub-bundles", etc.

$$\begin{array}{ll} SV & \\ \downarrow S\xi & = \text{unit sphere} \\ X & \quad \text{subbundle} \end{array}$$

$$\boxed{\eta = (\eta^{(1)}, \ldots, \eta^{(3k)}) \text{ codim-3, } \mathbb{Z}/2\text{-invariant}^{*} \text{ chains on } SV}$$

For $A \subseteq \{1, \ldots, 2k\}$ we have forgetful maps

$$\begin{tikzcd} \bar{S}_{2k}(V) \arrow[rr, "\varphi_A"] \arrow[dr] & & \bar{S}_2(V) = SV \arrow[dl] \\ & X & \end{tikzcd}$$

For a trivalent graph $\Gamma \in \mathrm{Graphs}^{\circ}_k$, define

$$\boxed{J(\xi, \eta, \Gamma) := \bigcap_{j=1}^{3k} \varphi^{-1}_{\Gamma(j)}(\eta^{(j)}) \in C_\ell(\bar{S}_{2k}(V))}$$

(assuming that the intersection is transverse).

Then define

$$\boxed{\alpha_k(\xi, \eta) := \frac{\lambda_k}{N'_k} \sum_{\Gamma \in \mathrm{Graphs}^{\circ}_k} J(\xi, \eta, \Gamma) \otimes [\Gamma] \in C_\ell(\bar{S}_{2k}(V)) \underset{\mathbb{Q}}{\otimes} \mathcal{A}_k}$$

N'_k = integer to be defined later

When $\ell = 0$ (i.e. $\dim(X) = k+5$):

$$\begin{array}{c} \alpha_k(\xi, \eta) \\ \cap \\ C_0(\bar{S}_{2k}(V)) \underset{\mathbb{Q}}{\otimes} \mathcal{A}_k \end{array} \xrightarrow{\mathrm{aug} \otimes \mathrm{id}} \mathbb{Q} \underset{\mathbb{Q}}{\otimes} \mathcal{A}_k = \mathcal{A}_k$$

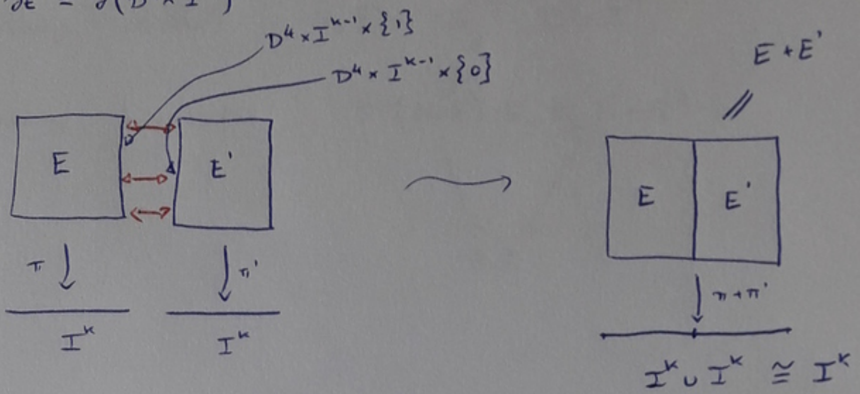
By abuse of notation, we denote the image also by $\boxed{\alpha_k(\xi, \eta) \in \mathcal{A}_k}$.

* In general, if a bundle has str group G, then its total space has a natural action of $Z(G)$. In this case $G = O(4)$, and $-I \in Z(O(4))$, so $-I$ generates a $\mathbb{Z}/2$-action

Construction of $\hat{Z}_k$:

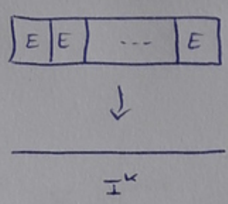
given $\begin{matrix} E & E' \\ \pi\downarrow & \swarrow\pi' \\ B = I^k & \end{matrix}$ smooth D^4-bundles with $\partial E = \partial E' = \partial(D^4 \times I^k)$

we may glue them :



This corresponds to addition in $[(I^k, \partial I^k), (BDiff_\partial(D^4), *)]$
$$\| $$
$$\pi_k BDiff_\partial(D^4)$$

For $n \geq 1$, let $nE = \overbrace{E + \cdots + E}^{n}$ =



"The n^{th} iteration of E"

Def : G finite abelian group $\longrightarrow m(G) := \min\{r \in \mathbb{N} \mid g^r = id \ \ \forall g \in G\}$

Thm : • Θ_n is finite for $n \geq 5$ [Smale + Kervaire-Milnor]

abelian group of homotopy n-spheres

h-cob. thm.

• $\Theta_n \cong \pi_0 Diff_\partial(D^{n-1})$ for $n \geq 6$ [Smale + Cerf]

Lemma : • $\pi_n O(4)$ is finite for $n \geq 4$ [proof uses fibration sequence $O(n) \hookrightarrow O(n+1) \twoheadrightarrow S^n$ and the fact [Serre] that $\pi_n S^2 \cong \pi_n S^3$ are finite for $n \geq 4$]

Def : $\boxed{N_k' := m(\Theta_{k+4}) \cdot m(\Theta_{k+5}) \cdot m(\pi_{k+3} O(4)) \cdot m(\pi_{k+4} O(4))}$.

$(k \geq 1)$

Prop: For any $\begin{array}{c} E \\ \pi\downarrow \\ I^k \end{array}$ we have:

(a) $\exists$ canonical-up-to-isotopy diffeo $N_k' E \cong_\partial D^4 \times I^k$

(b) $\exists$ unique-up-to-homotopy trivialisation

$$\begin{tikzcd} T^v(N_k' E) \arrow[rr, "\cong_\partial"] \arrow[dr] & & N_k' E \times \mathbb{R}^4 \arrow[dl] \\ & N_k' E & \end{tikzcd}$$

Proof: Set $n_1 = m(\Theta_{k+4})$

$n_2 = m(\Theta_{k+4}) \cdot m(\Theta_{k+5})$

$n_3 = m(\Theta_{k+4}) \cdot m(\Theta_{k+5}) \cdot m(\pi_{k+3} O(4)) \cdot m(\pi_{k+4} O(4)) = N_k'$.

(a) • Let $S^E := E \cup_\partial (D^4 \times I^k)$ (makes sense since $\partial E = \partial(D^4 \times I^k)$)

and note that $S^E \in \Theta_{k+4}$.

Hence $S^{n_1 E} \cong \underbrace{S^E \# \cdots \# S^E}_{n_1} \cong S^{k+4}$

$S^{n_1 E} = (n_1 E) \cup_\partial (D^4 \times I^k)$

$\Rightarrow \quad n_1 E \cong S^{k+4} - \text{int}(\text{some } (k+4)\text{-disc})$

$\cong S^{k+4} - \text{int}(\text{northern hemisphere})$ [Disc theorem & Palais]

$= D^{k+4}$.

• Any two such diffeo's differ (up to $\simeq$) by an element of $\pi_0 \text{Diff}_\partial(D^{k+4}) \cong \Theta_{k+5}$

$\Rightarrow$ iterating such a diffeo $m(\Theta_{k+5})$ times, it becomes <u>unique</u> up to isotopy.

$\Rightarrow \exists$ unique-up-to-isotopy diffeo $n_2 E \cong_\partial D^4 \times I^k$ that arises as an iteration of a diffeo $n_1 E \cong_\partial D^4 \times I^k$.

(b) obstruction theory:

- obstructions to trivialising $\begin{array}{c} T^v(n_2 E) \\ \downarrow \\ n_2 E \end{array}$ lie in $H^*(n_2 E, \partial n_2 E ; \pi_{*-1} O(4))$

- if $\exists$, obstructions to uniqueness (up to $\simeq$) lie in $H^*(n_2 E, \partial n_2 E ; \pi_* O(4))$

By (a), $\boxed{n_2 E \cong_\partial D^{k+4}}$ so we just need

$$0 \overset{?}{=} \text{obs} \in H^{k+4}(n_2 E, \partial n_2 E ; \pi_{k+3} O(4))$$

$$0 \overset{?}{=} \text{obs} \in H^{k+4}(n_2 E, \partial n_2 E ; \pi_{k+4} O(4))$$

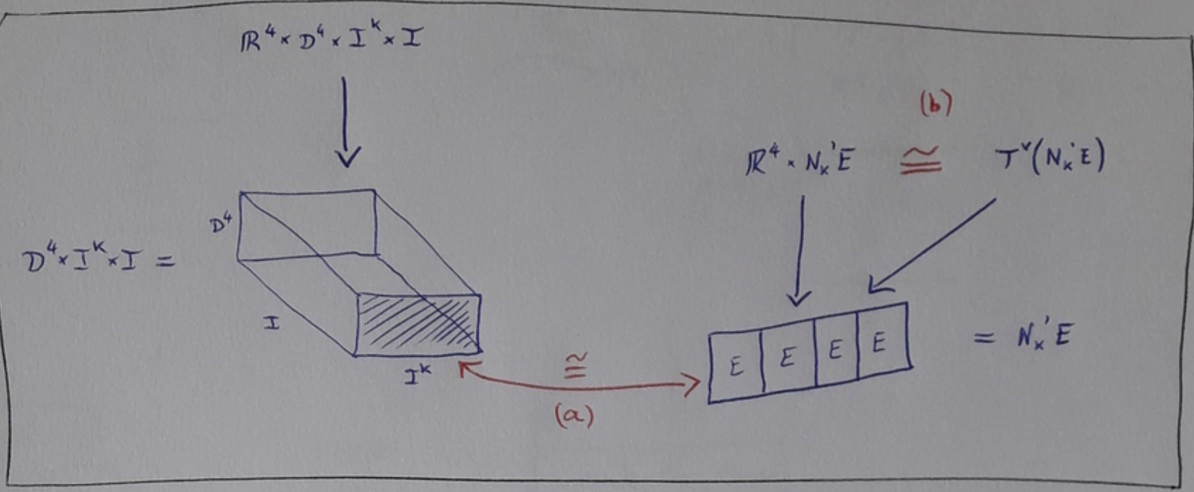
Iteration of bundles $\longrightarrow$ addition of the obstructions using the group structure of $\pi_* O(4)$

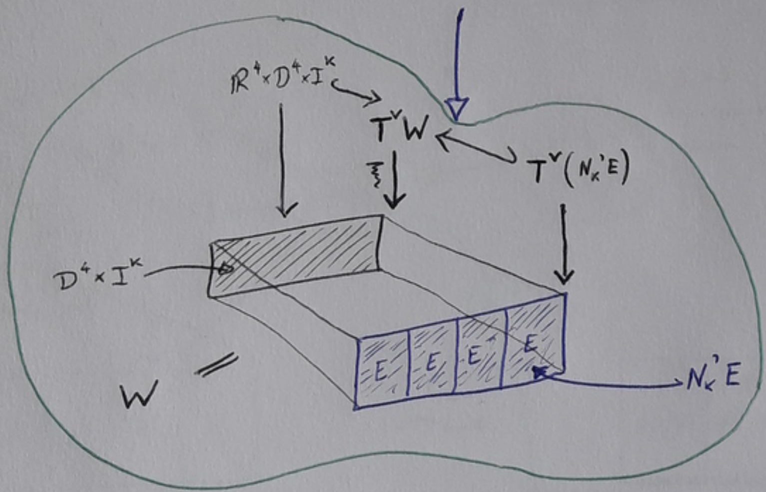
$\Rightarrow$ iterating by $m(\pi_{k+3} O(4)) \cdot m(\pi_{k+4} O(4)) \longrightarrow$ done. $\square$.

Construction of $\hat{Z}_k$:

Step ①

$$\begin{array}{c} E \\ {\scriptstyle \pi}\downarrow \\ I^k \end{array} \quad \begin{array}{l}\text{smooth } D^4\text{-bundle} \\ \text{with } \partial E \text{ trivial}\end{array} \quad \longrightarrow \quad \begin{array}{c} T^vW \\ {\scriptstyle \xi}\downarrow \\ W^{k+5} \end{array} \quad \mathbb{R}^4\text{- bundle}$$





Note: W is a bordism (rel. ∂) of $(k+4)$-manifolds with $\mathbb{R}^4$-bundles

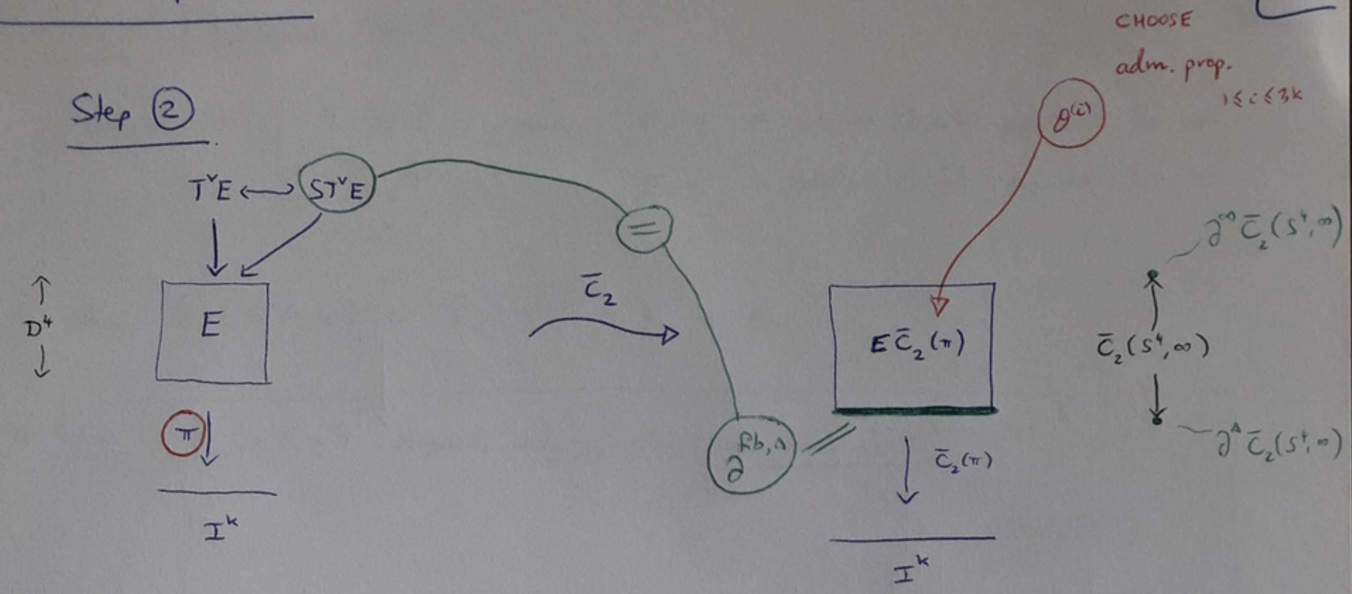
$$\begin{array}{c} \mathbb{R}^4\times D^4\times I^k \\ \downarrow \\ D^4\times I^k \end{array} \quad \rightsquigarrow \quad \begin{array}{c} T^v(N_k'E) \\ \downarrow \\ N_k'E \end{array}$$

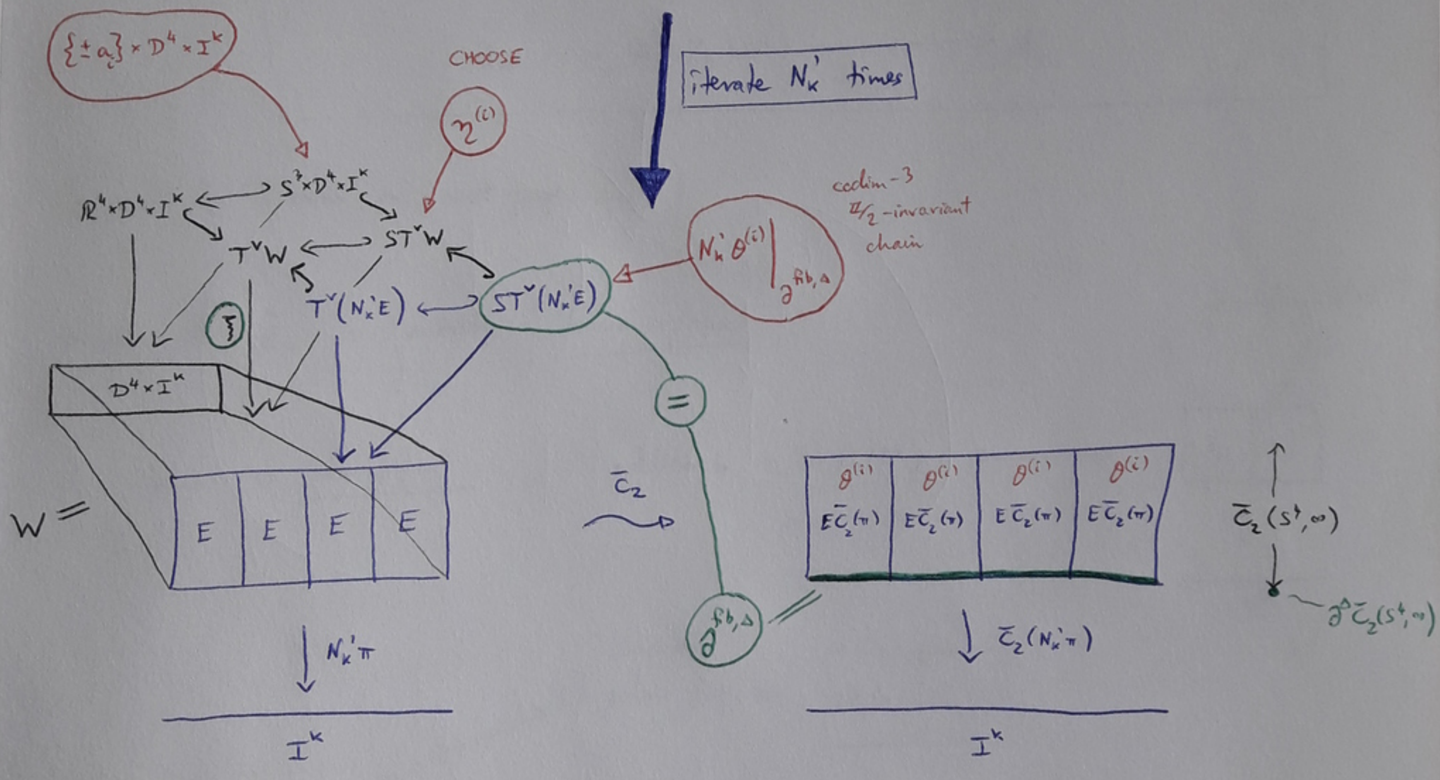
but <u>not</u> (!) a bordism of <u>smooth D^4-bundles</u>.

This is because (a) is a diffeo. (rel. ∂) of <u>manifolds</u>, not of <u>smooth D^4-bundles</u>.

Construction of $\hat{Z}_k$:

Step ②





$\eta^{(i)}$ codim-3, $\mathbb{Z}/2$-invariant chain such that

$$\partial \eta^{(i)}\Big|_{\text{front face}} = N_k' \theta^{(i)}\Big|_{\partial^{fb,\Delta}}$$

$$\partial \eta^{(i)}\Big|_{\text{back face}} = \{\pm a_i\} \times D^4 \times I^k \qquad (a_i \in S^3)$$

Def: $\boxed{\hat{Z}_k(\pi, \theta, \eta) := Z_k(\pi, \theta) - \alpha_k(\xi, \eta) \quad \in \mathcal{A}_k}$

Theorem [Watanabe, Thm 2.7]

suppose that
$$\begin{tikzcd} E \arrow[rr, phantom, "\cong_\partial"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & I^k & \end{tikzcd}$$
and θ, η = choices of adm. prop. etc for π

θ', η' = choices of adm. prop. etc for π'

then $\hat{Z}_k(\pi, \theta, \eta) = \hat{Z}_k(\pi', \theta', \eta') \in \mathcal{A}_k$.

↳ Coro

> $\hat{Z}_k(\pi, \theta, \eta)$ depends only on the isom. class of π $\xleftrightarrow{1:1}$ $\pi_k BDiff_\partial(D^4)$
>
> Thus we get a well-defined function
>
> $$\hat{Z}_k : \pi_k BDiff_\partial(D^4) \longrightarrow \mathcal{A}_k$$

(proof of Theorem on next pages ......)

Lemma: $\hat{Z}_k$ is actually a homomorphism

↳ idea of proof:

(i) addition in $\pi_k BDiff_\partial(D^4) \ni \alpha + \beta$ $\longleftrightarrow$ $\begin{array}{c} \boxed{E_\alpha \mid E_\beta} \\ \downarrow \\ I^k \end{array}$ glue along $D^4 \times I^{k-1}$

(ii) choice of adm. prop. on $\boxed{E\bar{C}_2(\pi_\alpha) \mid E\bar{C}_2(\pi_\beta)}$

restricts to an adm prop on $E\bar{C}_2(\pi_\alpha)$ and on $E\bar{C}_2(\pi_\beta)$.

NB: conversely, if
θ_α adm prop on $E\bar{C}_2(\pi_\alpha)$
θ_β -------- $E\bar{C}_2(\pi_\beta)$
then $\theta_\alpha + \theta_\beta$ adm prop on $\boxed{E\bar{C}_2(\pi_\alpha) \mid E\bar{C}_2(\pi_\beta)}$ F

$\Longleftrightarrow \partial\theta_\alpha|_F = -\partial\theta_\beta|_F$

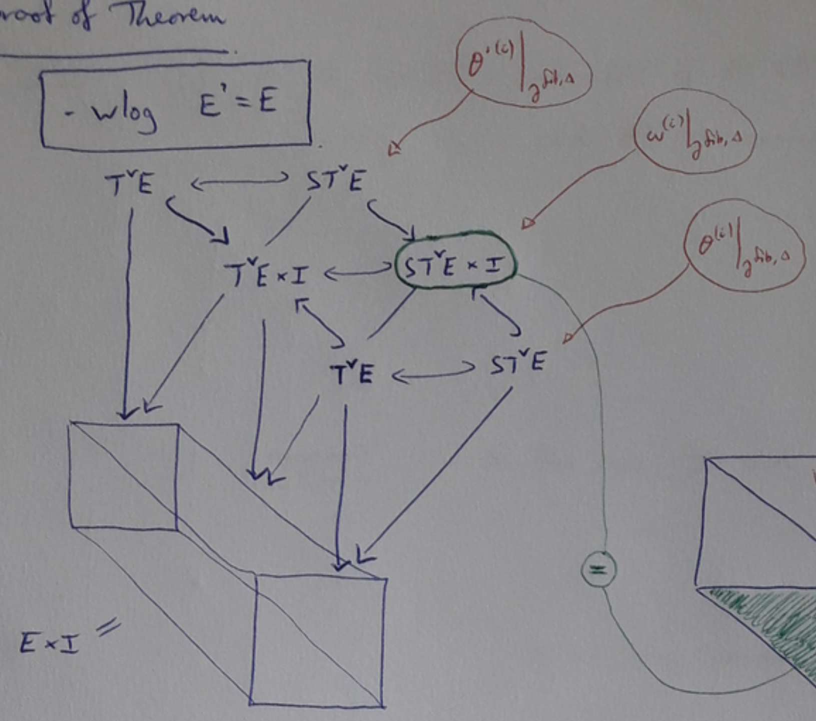
(iii) $\hat{Z}_k(\alpha + \beta) = \sum$ intersections of pullbacks of adm. prop.'s on $\boxed{}\boxed{}$

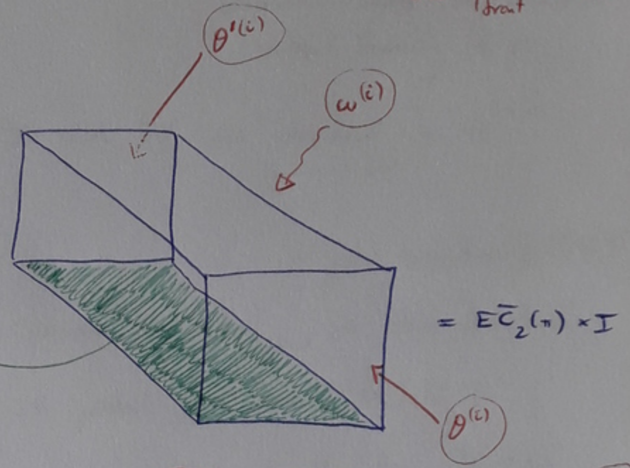
$= \underbrace{\sum \text{intersections .... on } \boxed{}\,\square}_{\hat{Z}_k(\alpha)} + \underbrace{\sum \text{intersection ... on } \square\,\boxed{}}_{\hat{Z}_k(\beta)}$

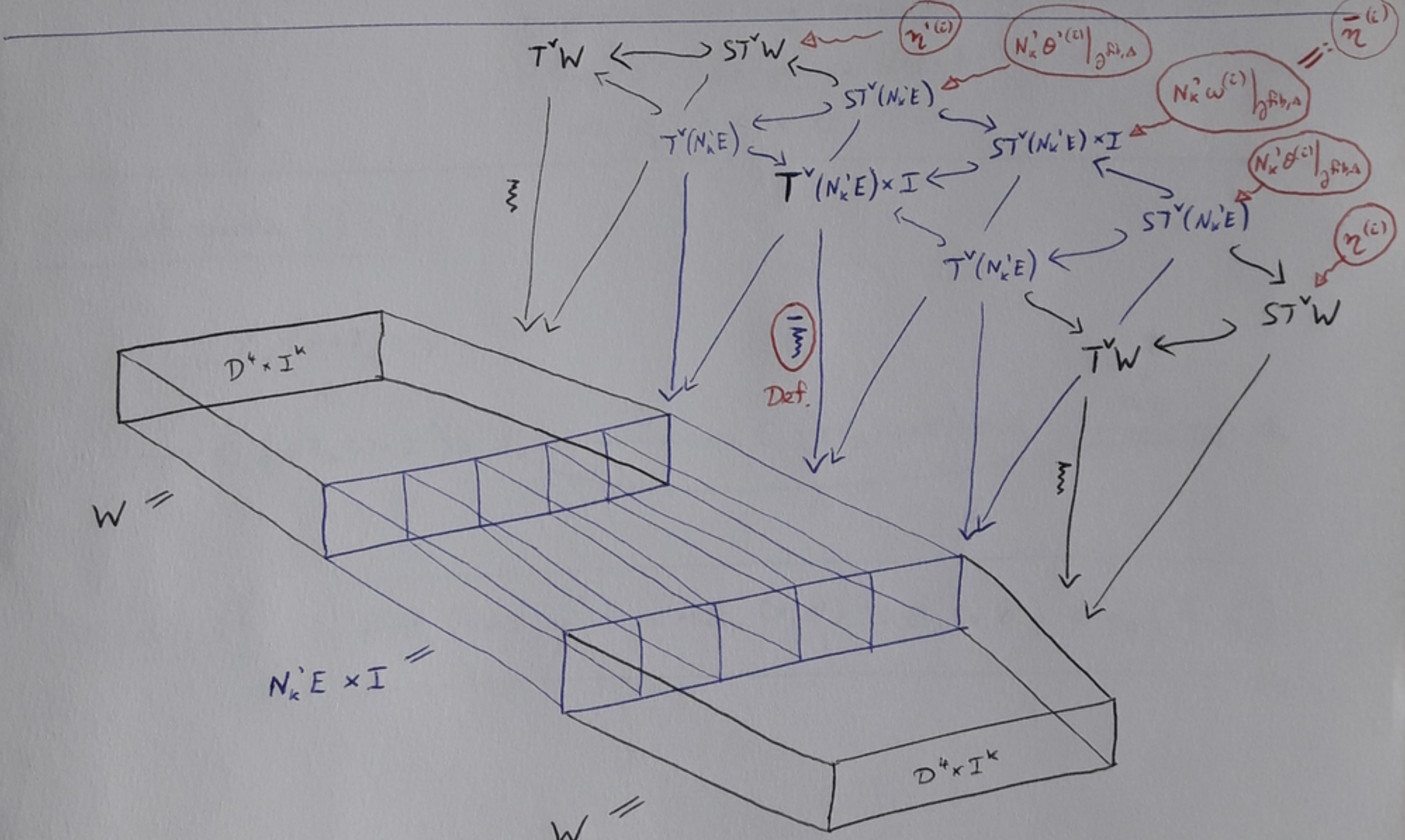
proof of Theorem

$\boxed{\text{- wlog } E' = E}$

Lem: $\exists$ adm. prop. $\omega^{(i)}$ on $E\bar{C}_2(\pi)\times I$ with
$\partial\omega^{(i)}|_{back} = \theta'^{(i)}$
$\partial\omega^{(i)}|_{front} = -\theta^{(i)}$







Claim: (1) $Z_k(\pi,\theta) - Z_k(\pi,\theta') = \alpha_k(\bar{\xi},\bar{\eta})$

(2) $\alpha_k(\xi,\eta) - \alpha_k(\xi,\eta') = \alpha_k(\bar{\xi},\bar{\eta})$

Note that
(1) - (2)
$\Downarrow$
Theorem.

Aside: (1) $\Rightarrow$ $Z_k(\pi,\theta)$ is independent of $\theta = (\theta^{(1)} \dots \theta^{(3k)})$ <u>if</u>

each $\theta^{(i)}$ is assumed to be a $\underbrace{\text{strong adm. prop.}}$

depends on choice (in particular, $\exists$) vertical framing of π

argument: - in this case, $\partial\theta$ and $\partial\theta'$ are $\underbrace{\text{"standard"}}$ on $\partial^{fib,\Delta}$

pulled back from $\{\pm a\} \subseteq S^3$

& we may assume that $\partial\omega|_{\partial^{fib,\Delta}}$ is also standard

- $\Rightarrow$ $\bar{\eta}^{(i)}$ are all pulled back from $\{\pm a_i\} \subseteq S^3$
- generically, we may assume that $a_i \neq \pm a_j$ for $i \neq j$

$\Rightarrow$ $\bar{\eta}^{(i)}$ are pairwise disjoint

$\Rightarrow$ $\alpha_k(\bar{\xi}, \bar{\eta}) = 0.$

Proof of equation (1):

$$
\begin{array}{c}
Z_k(\pi \times I, \omega) \\
\cap \\
\cdots \longrightarrow C_1(E\bar{C}_{2k}(\pi) \times I) \otimes A_k \xrightarrow{\ \partial\ } C_0(E\bar{C}_{2k}(0) \times I) \otimes A_k \xrightarrow[\text{aug.}]{\#} A_k
\end{array}
$$

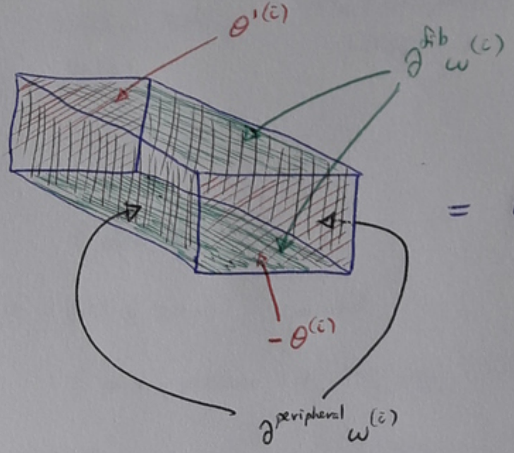
0

Claim (1)': $\boxed{\#\partial Z_k(\pi \times I, \omega) = -Z_k(\pi,\theta) + Z_k(\pi,\theta') + \alpha_k(\bar{\xi},\bar{\eta})}$

$$\# \partial Z_k(\pi \times I, \omega) = \lambda_k \sum_{\Gamma} \underbrace{\partial I(\pi \times I, \omega, \Gamma)}\,.\,[\Gamma] \quad \in \mathcal{A}_k$$

$$\partial I(\pi \times I, \omega, \Gamma) = \sum_{\substack{X \subseteq \{1, \dots, 3k\} \\ |X| \geq 1}} I\left(\pi \times I, \left\{ \begin{matrix} \omega^{(i)} & i \notin X \\ \underbrace{\partial \omega^{(i)}} & i \in X \end{matrix} \right\}, \Gamma\right)$$

$$\partial \omega^{(i)} = \theta'^{(i)} - \theta^{(i)} + \partial^{fib} \omega^{(i)} + \partial^{peripheral} \omega^{(i)} \qquad (*)$$



$$= E\bar{C}_2(\pi) \times I$$

Hence $\# \partial Z_k(\pi \times I, \omega)$

$= \sum$ terms involving $\theta'^{(i)}$ $\longleftrightarrow$ $Z_k(\pi, \theta')$

$+ \sum$ terms involving $\theta^{(i)}$ $\longleftrightarrow$ $-Z_k(\pi, \theta)$

$+ \sum$ terms involving $\partial^{peripheral} \omega^{(i)}$ $\longleftrightarrow$ 0 — since $\partial^{peripheral} \omega^{(i)}$ is supported in the faces of $E\bar{C}_2(\pi) \times I$ corresponding to a trivial D^4-bundle over $\partial I^k \times I$

$+ \sum$ terms involving $\partial^{fib} \omega^{(i)}$ $\longleftrightarrow$ ....

+ no mixed terms — since the terms of $(*)$ have support in different faces of $E\bar{C}_2(\pi) \times I$.

$$\partial^{fib}\omega^{(i)} = \sum_{\substack{A\subseteq\{1\dots 2k,\infty\}\\ |A|\geqslant 2}} \underbrace{\partial^{fib,A}\omega^{(i)}}$$

the piece supported in the fibrewise $-\partial-$ stratum of $E\bar{C}_2(\pi)\times I$ corresponding to A.

$$\sum \text{terms involving } \partial^{fib}\omega^{(i)} = \lambda_k \sum_{\Gamma}\sum_{X}\sum_{A} I\left(\pi\times I, \left\{\begin{matrix}\omega^{(i)} & i\notin X\\ \partial^{fib,A}\omega^{(i)} & i\in X\end{matrix}\right\}, \Gamma\right).[\Gamma]$$

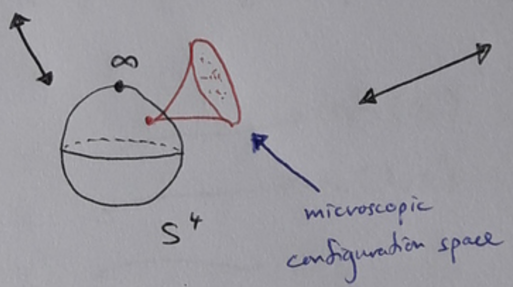
Γ: trivalent graph, vertices $=\{1\dots 2k\}$, edges $=\{1\dots 3k\}$

X: $X\subseteq\{1\dots 3k\}$, $|X|\geqslant 1$

A: $A\subseteq\{1,\dots,2k,\infty\}$, $|A|\geqslant 2$

Facts:

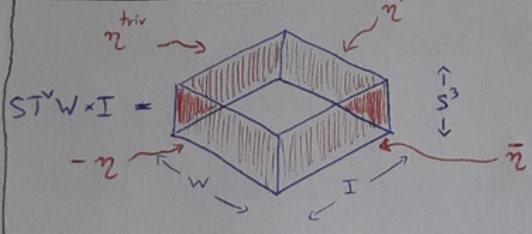
- terms with $\infty\in A$ cancel (dim. reasons)
- terms with $\infty\notin A$ and $3\leqslant |A|\leqslant 2k-1$ cancel (dim. reasons + $\mathbb{Z}/2$-symmetry)
- terms with $\infty\notin A$ and $|A|=2$ cancel (IHX relations)
- terms with $A=\{1,\dots,2k\}$



$\alpha_k(\bar{\xi},\bar{\eta})$.

Proof of equation (2) : (philosophically similar)

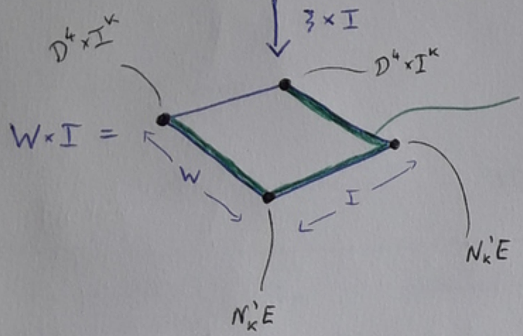
$$
\begin{tikzcd}
\alpha_k(\xi \times I, \bar{\bar{\eta}}) \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & & \\
C_1(ST^vW \times I) \otimes \mathcal{A}_k \arrow[r, "\partial"] \arrow[rr, bend right, "0"'] & C_0(ST^vW \times I) \otimes \mathcal{A} \arrow[r, "\underset{\text{aug.}}{\#}"] & \mathcal{A}_k
\end{tikzcd}
$$



Lem: $\exists\ \bar{\bar{\eta}}$ $\mathbb{Z}/_2$-inv. codim. 3-chain on $ST^vW \times I$ such $\partial\bar{\bar{\eta}}$ restricts to η, η^{triv}, η' and $\bar{\eta}$

$\eta^{triv} = \{\pm a\} \times D^4 \times I^{k+1}$ on $S^3 \times D^4 \times I^{k+1}$

$\downarrow \xi \times I$



= diagram at bottom of page 12 (with 3 dim. collapsed to 1 dim.)

Claim (2)': $\boxed{\#\partial\alpha_k(\xi \times I, \bar{\bar{\eta}}) = -\alpha_k(\xi, \eta) + \alpha_k(\xi, \eta') + \alpha_k(\bar{\xi}, \bar{\eta})}$

Note that $\quad \partial\bar{\bar{\eta}} = \eta' - \eta + \eta^{triv} + \bar{\eta} + \partial^{fib}\bar{\bar{\eta}} \qquad (**)$

Hence $\#\partial\alpha_k(\xi \times I, \bar{\bar{\eta}})$

$= \sum$ terms involving η' $\longrightarrow \alpha_k(\xi, \eta')$

$+ \sum$ terms involving η $\longrightarrow -\alpha_k(\xi, \eta)$

$+ \sum$ terms involving η^{triv} $\longrightarrow 0$

$+ \sum$ terms involving $\bar{\eta}$ $\longrightarrow \alpha_k(\bar{\xi}, \bar{\eta})$

$+ \sum$ terms involving $\partial^{fib}\bar{\bar{\eta}}$

$\longrightarrow$ all strata except $A = \{1, 2, \ldots, 2k\}$ cancel as before (dim. reasons + $\mathbb{Z}/_2$-symmetry + IHX relation)

This time $\nexists$ stratum corresponding to $A = \{1, 2, \ldots, 2k\}$ in $\bar{S}_{2k}$, since <u>not all</u> points may collide simultaneously in microscopic config spaces.

Oriented bordism-invariance of $\hat{Z}_k$

Oriented bordism:

$$\tilde{\Omega}_k(X) = \{\text{closed } k\text{-manifolds with based map to } X\} \big/ \text{oriented bordism}$$

$$\rightsquigarrow \tilde{\Omega}_k(BG) \cong \{\text{closed } k\text{-manifolds with principal } G\text{-bundle trivialised over the basepoint}\} \big/ \text{oriented bordism}$$

$$\rightsquigarrow \tilde{\Omega}_k(B\mathrm{Diff}_\partial(D^4)) \cong \{\text{closed } k\text{-manifolds with smooth } D^4\text{-bundle (with trivial } \partial\text{) trivialised over the basepoint}\} \big/ \text{oriented bordism}$$

$\exists$ natural map

$$H = \text{"Hurewicz"} : \pi_k B\mathrm{Diff}_\partial(D^4) \longrightarrow \tilde{\Omega}_k(B\mathrm{Diff}_\partial(D^4))$$

Proposition: $\hat{Z}_k$ is oriented-bordism-invariant, in other words:

$$\begin{tikzcd}
\pi_k B\mathrm{Diff}_\partial(D^4) \arrow[rr, "H"] \arrow[dr, two heads] \arrow[dd, "\hat{Z}_k"'] & & \tilde{\Omega}_k(B\mathrm{Diff}_\partial(D^4)) \\
& \mathrm{im}(H) \arrow[ur, hook] \arrow[dl, dashed] & \\
A_k & &
\end{tikzcd}$$

More generally, if $\tilde{h}_*$ is a (reduced) generalised homology theory, choosing $x \in \tilde{h}_0(S^0) = h_0(pt)$ determines a Hurewicz map

$$\tilde{h}_0(S^0) \cong \tilde{h}_k(S^k)$$

$$\pi_k(X) \xrightarrow{H} \tilde{h}_k(X)$$

$$[S^k \xrightarrow{f} X] \longmapsto f_*(x)$$

Proposition': if $x \in h_0(pt)$ generates a $\mathbb{Z}$-summand then:

$$\begin{tikzcd}
\pi_k(BG) \arrow[rr, "H"] \arrow[dr, two heads] \arrow[dd, "\varphi"'] & & \tilde{h}_k(BG) \\
& \mathrm{im}(H) \arrow[ur, hook] \arrow[dl, dashed] & \\
V & &
\end{tikzcd}$$

φ: any homom. to a $\mathbb{Q}$-vector-space

G any topological group

Prop' ⇒ Prop : take $G = \mathrm{Diff}_\partial(D^4)$
$\varphi = \hat{Z}_K$
$\tilde{h}_* =$ oriented bordism

and note that $\Omega_0(pt) \cong \mathbb{Z}\langle \cdot^+ \rangle$ □.

Rmk : - We could take $\tilde{h}_* =$ K-theory
- But we cannot take $\tilde{h}_* =$ unoriented bordism, since $\Omega_0^{unor}(pt) \cong \mathbb{Z}/2\langle \cdot \rangle$

---

sketch proof of Prop' :

$$
\begin{tikzcd}
\pi_K BG \arrow[rr, "H"] \arrow[dr] \arrow[dd, "\varphi"'] & & \tilde{h}_K(BG) \arrow[dr] & \\
& \pi_K BG \otimes \mathbb{Q} \arrow[rr, "H\otimes\mathbb{Q}"] \arrow[dl] & & \tilde{h}_K(BG)\otimes\mathbb{Q} \\
V & & &
\end{tikzcd}
$$

- By a diagram chase, it suffices to prove that **$H\otimes\mathbb{Q}$ is injective**.
- To see this:

class.^{n} of rational gen. hom. theories

$$
\begin{tikzcd}
\pi_* BG \otimes \mathbb{Q} \arrow[r, "H\otimes\mathbb{Q}"] \arrow[dd, hook] & \tilde{h}_*(BG)\otimes\mathbb{Q} \arrow[r, phantom, "\cong"] & \tilde{H}_*(BG;\mathbb{Q})\otimes h_*(pt) \\
& \circlearrowleft & H_*(BG;\mathbb{Q})\otimes h_*(pt) \arrow[u, "(\cong \text{ in deg}>0)\otimes \mathrm{id}"'] \\
\mathcal{U}(\pi_* BG\otimes\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[rr, "\mathrm{id}\otimes(1\mapsto x)"] & & \mathcal{U}(\pi_* BG\otimes\mathbb{Q})\otimes h_*(pt) \arrow[u, "\cong"', "\text{Milnor-Moore}"]
\end{tikzcd}
$$

split-injective iff $x \in h_0(pt)$ generates a $\mathbb{Z}$-summand.

universal enveloping Hopf algebra

□.