Kontsevich characteristic classes à la Watanabe

(I)

GeMAT seminar

IMAR

12 June 2020

Following: T. Watanabe

"Some exotic nontrivial elements of the rational homotopy groups of $\mathrm{Diff}(S^4)$"

arXiv: 1812.02448

Goal: "exotic" elements of $\pi_* \mathrm{Diff}(S^4) \otimes \mathbb{Q}$

not in the image of $\pi_* SO(5) \otimes \mathbb{Q} \longrightarrow \pi_* \mathrm{Diff}(S^4) \otimes \mathbb{Q}$
induced by $SO(5) \hookrightarrow \mathrm{Diff}(S^4)$

elements of $\pi_* \mathrm{Diff}_\partial(D^4) \otimes \mathbb{Q}$

Plan: (I) + (II) Define invariant

$$\hat{Z}_k : \underbrace{\pi_{k-1} \mathrm{Diff}_\partial(D^4) \otimes \mathbb{Q}}_{\{D^4\text{-bundles over } S^k\}} \longrightarrow \underbrace{\mathcal{A}_k}_{\text{graph complex homology}}$$

(III) Construct candidate D^4-bundles E^Γ

(IV) + ... Prove that $\hat{Z}_k(E^\Gamma) = [\Gamma] \in \mathcal{A}_k$, $[\Gamma] \neq 0$

k	$\pi_k SO(5) \otimes \mathbb{Q}$
1	0
2	0
3	$\mathbb{Q}$
4	0
5	0
6	0
7	$\mathbb{Q}$
8	0
⋮	⋮
15	0

Kontsevich characteristic classes à la Watanabe

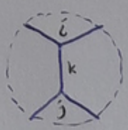
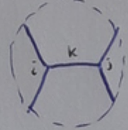
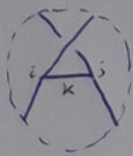
①

Trivalent graphs

ref: [W, §2.1]

Definition $\mathcal{A}_k := \mathbb{Q}\left\{ \begin{array}{l} \text{connected trivalent graphs with } 2k \text{ vertices (hence } 3k \text{ edges)} \\ \text{with } \left[\begin{array}{l}\text{vertices labelled by } \{1 \ldots 2k\} \\ \text{edges labelled by } \{1 \ldots 3k\}\end{array}\right. \end{array} \right\} \Big/ \sim$

where $\sim$ is generated by

(IHX) (i)  +  +  ~ 0

(ii) $\Gamma + \Gamma' \sim 0$

exchange labels of two edges

[(iii) $\Gamma - \Gamma'' \sim 0$

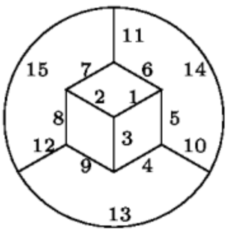
exchange labels of two vertices]

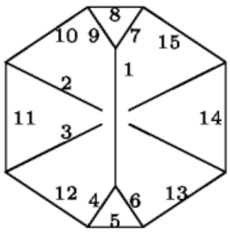
Fact: We have:

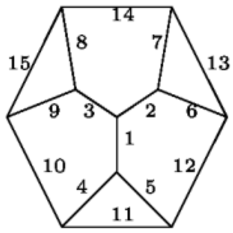
k	1	2	3	4	5	6	7	8	9
$\dim_{\mathbb{Q}}(\mathcal{A}_k)$	0	1	0	0	1	0	0	0	1

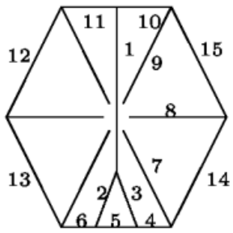
P.T.O.

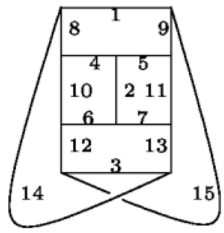
[Bar-Natan, McKay]

$\frac{2}{3}$ 

$+$ 

$+\frac{4}{3}$ 

$+2$ 

$+$ 

Larger context:

Graph complex $\quad \mathcal{G} = \bigoplus_{d,e\geq 0} \mathcal{G}_d^e$

$$\mathcal{G}_d^e := \mathbb{Q}\left\{ \text{all graphs with } \begin{array}{l} \#E - \#V = \overset{\text{degree}}{d} \\ 2\#E - 3.\#V = \underset{\text{excess}}{e} \end{array} \right\}$$

no multiple edges
no loops
all vertices of valence ≥ 3

(note: trivalent $\Longleftrightarrow$ excess $= 0$)

This has a differential $\mathcal{G}_d^e \longrightarrow \mathcal{G}_d^{e-1}$ given by edge contraction.

Remark: $\quad H^0_k(\mathcal{G}) = \ker\left(\mathcal{G}_k^0 \longrightarrow 0\right) \Big/ \mathrm{im}\left(\mathcal{G}_k^1 \longrightarrow \mathcal{G}_k^0\right)$

(IHX)

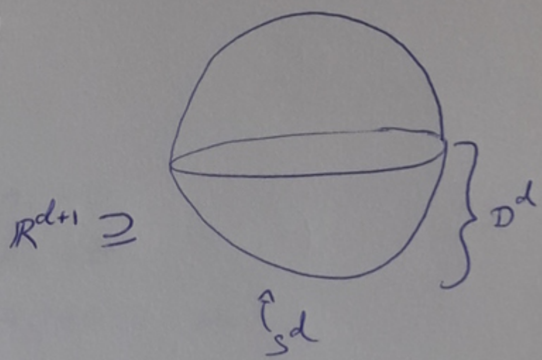
$$\cong \mathcal{A}_k$$

Kontsevich char. classes more generally take values in $H^*_*(\mathcal{G})$,
but we (and Watanabe) just need $H^0_*(\mathcal{G}) \cong \mathcal{A}_*$.

Aim: Understand $\pi_* \mathrm{Diff}(S^4)$

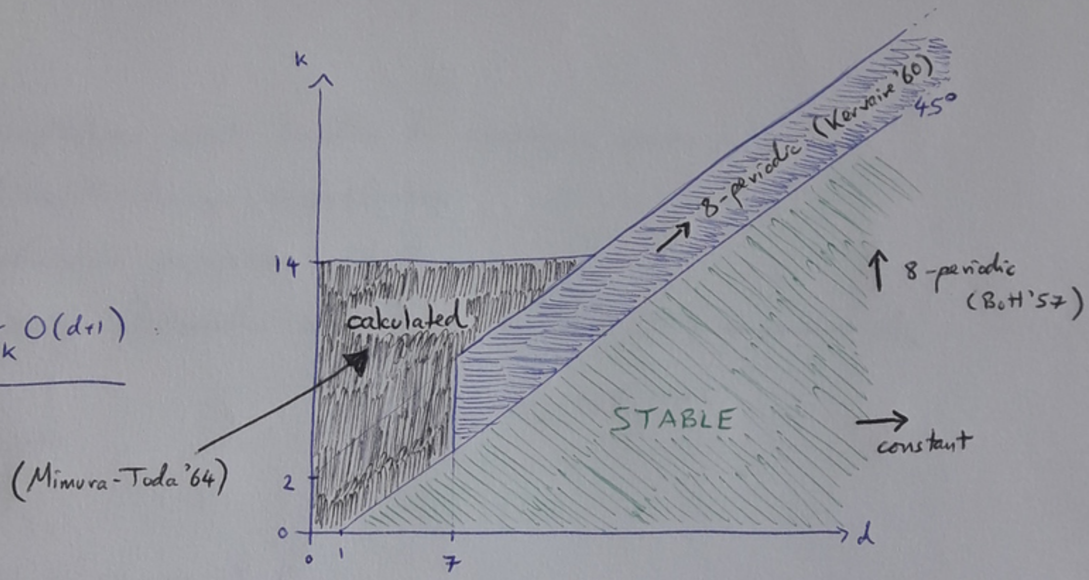
In general:

$$\begin{tikzcd}
\mathrm{Diff}_\partial(D^d) \arrow[dr, hook, "\text{extend by id.}"] & \\
& \mathrm{Diff}(S^d) \\
O(d+1) \arrow[ur, hook, "\text{restrict}"'] &
\end{tikzcd}$$



Fact: $\mathrm{Diff}_\partial(D^d) \times O(d+1) \xrightarrow{\simeq} \mathrm{Diff}(S^d)$

$\pi_k O(d+1)$:



$\underline{d \leq 3}$ $\quad \mathrm{Diff}_\partial(D^d) \simeq *$ $\quad$ (Smale '59, Hatcher '83)
$\qquad\qquad d=2 \qquad\qquad d=3$

$\pi_k \mathrm{Diff}_\partial(D^d)$:

$\underline{d \geq 5}$ $\quad \pi_0 \mathrm{Diff}_\partial(D^d) \cong \Theta_{d+1} = \{\text{homotopy } (d+1)\text{-spheres}\}/\text{diffeo}^+$

(Smale '62 + Cerf '70)
(surj) $\qquad$ (inj)

- finite [Kervaire-Milnor '63]
- $\neq 0$ for all $d \in \{6, 7, \ldots, 61\}$ except 11, 55 and 60 and all <u>even</u> $d \geq 62$.

$\underline{d=4}$ (Watanabe '18):

$$\begin{tikzcd} \pi_{k-1}\mathrm{Diff}_\partial(D^4) \arrow[rr, "\hat{Z}_k"] \arrow[dr] & & A_k \\ & \pi_{k-1}\mathrm{Diff}_\partial(D^4)\otimes\mathbb{Q} \arrow[ur, two heads, "\text{surjective}"'] & \end{tikzcd}$$

Corollary: $\pi_1 \mathrm{Diff}_\partial(D^4) \neq 0$ since $A_2 \neq 0$
$\pi_4 \mathrm{Diff}_\partial(D^4) \neq 0$ since $A_5 \neq 0$
$\pi_8 \mathrm{Diff}_\partial(D^4) \neq 0$ since $A_9 \neq 0$ etc.

Outline

- recollections about bundles & classifying spaces
- Fulton-MacPherson compactifications
- admissible propagators and Z_k
- iteration of bundles and $\hat{Z}_k : \pi_k B\mathrm{Diff}_\partial(D^4) \longrightarrow A_k$
  $\qquad\qquad\qquad\qquad\qquad\qquad \| $
  $\qquad\qquad\qquad\qquad\qquad \pi_{k-1}\mathrm{Diff}_\partial(D^4)$

next week

- well-defined-ness of $\hat{Z}_k$
- why it factors through image$\left(\pi_k B\mathrm{Diff}_\partial(D^4) \xrightarrow{H} \Omega_k(B\mathrm{Diff}_\partial(D^4))\right)$
- more formulas for $\hat{Z}_k$ on image(H)...

oriented bordism group

② Bundles and associated principal bundles

ref: Steenrod
"The topology of fibre bundles"
(1951)

"Def" $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ fibre bundle with fibre F
& structure group $G \curvearrowright F$

- locally looks like $\begin{matrix} F \times U \\ \downarrow \\ U \end{matrix}$ for $U \subseteq X$
- transition functions act on F through G.

Eg
- $GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$ — vector bundles
- $\mathrm{Diff}(M) \curvearrowright M$ — smooth M-bundles
- $F = G \curvearrowright G$ — principal G-bundles

Lem: If $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ is a principal G-bundle, there is an action of G on E,
and $X = E/G$

moreover:

$$\{\text{principal } G\text{-bundles}/X\}/_{\cong} \xrightarrow[(*)]{\text{Borel construction}} \{\text{fibre bundles}/X \text{ with fibre } F \text{ and str. group } G\}/_{\cong}$$

$$\begin{matrix} E \\ \downarrow \\ X \end{matrix} \longmapsto \begin{matrix} E \times_G F = (E \times F)/G \\ \downarrow \\ X \end{matrix}$$

Prop: The Borel construction is a bijection.

Def: The pre-image of a fibre bundle under $(*)$ is its associated principal bundle.

3 Bundles and classifying spaces

ref: Milnor
"Construction of universal bundles II"
(1956)

Classifying spaces

topological group $G \rightsquigarrow$

$$\begin{array}{c} EG \\ \downarrow \\ BG \end{array}$$

principal G-bundle

based space

Thm: $[X, BG] \xrightarrow{\cong} \{\text{principal } G\text{-bundles over } X\}/_{\cong}$ (X paracompact)

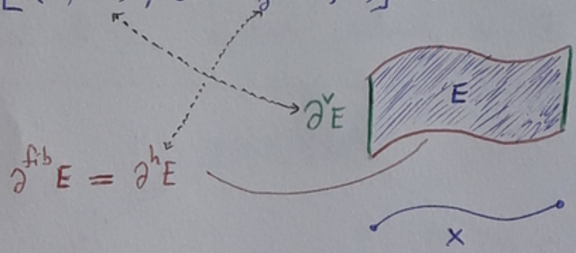
$$[f : X \to BG] \longmapsto \begin{array}{c} f^*(EG) \\ \downarrow \\ X \end{array}$$

Lemma: $\Omega(BG) \simeq G$

hence $\pi_{k-1}(G) \cong \pi_k(BG) \quad \forall k$

Cor: M, X manifolds

$$[(X, \partial X), (B\mathrm{Diff}_\partial(M), *)] \xleftrightarrow{\cong} \left\{ \begin{array}{l} \text{smooth } M\text{-bundles } \begin{array}{c} E \\ \downarrow \\ X \end{array} \\ \text{with } \partial E = \partial(M \times X) \end{array} \right\} /_{\cong}$$



Special case: $X = [0,1]^k = I^k$
$M = D^d$

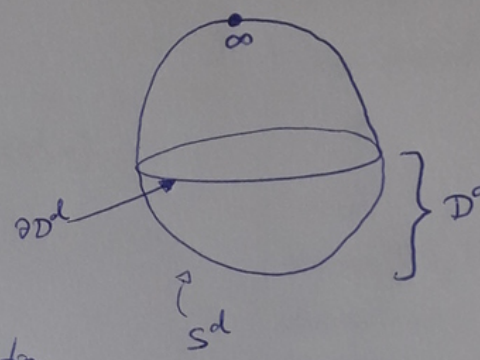
$$\begin{array}{c} \pi_k B\mathrm{Diff}_\partial(D^d) \\ \text{\rotatebox{90}{$\cong$}} \\ \pi_{k-1} \mathrm{Diff}_\partial(D^d) \end{array} \cong \left\{ \begin{array}{l} \text{smooth } D^d\text{-bundles } \begin{array}{c} E \\ \downarrow \\ I^k \end{array} \\ \text{with } \partial E = \partial(D^d \times I^k) \end{array} \right\} /_{\cong}$$

④ Fulton - MacPherson compactifications and the map $\partial\bar{C}_2(S^d,\infty) \to S^{d-1}$

ref: [W, §2.2, §2.3]

Def $C_n(S^d,\infty)$ = { ordered configurations of n distinct points in $S^d - \{\infty\}$ }

$$\begin{array}{c} C_n(S^d,\infty) \\ \circlearrowleft \\ \mathrm{Diff}_\partial(D^d) \\ \big\downarrow \\ \bar{C}_n(S^d,\infty) \end{array}$$



$\bar{C}_n(S^d,\infty)$ = its Fulton - MacPherson compactification

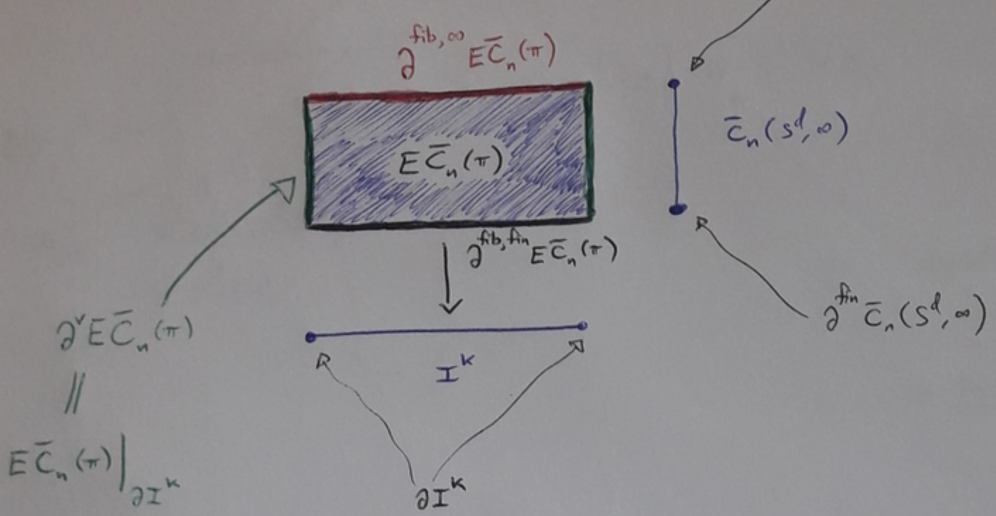
Key properties

- smooth compact manifold with corners
  - boundary strata $\longleftrightarrow$ subsets $A \subseteq \{1,2,\dots,n,\infty\}$, $|A| \geq 2$, $\phi \not\ni$
- its interior is $C_n(S^d,\infty)$
- the inclusion $C_n(S^d,\infty) \longrightarrow \bar{C}_n(S^d,\infty)$ is a homotopy equivalence
- the action of $\mathrm{Diff}_\partial(D^d)$ extends naturally to $\bar{C}_n(S^d,\infty)$
- the action of Σ_n extends naturally to $\bar{C}_n(S^d,\infty)$

$$\begin{tikzcd}[column sep=small] & \pi_k BDiff_\partial(D^d) \arrow[dl, leftrightarrow, "1:1"'] \arrow[dr, leftrightarrow, "1:1"] & \\ \left\{ \begin{array}{l} \text{smooth } D^d\text{-bundles } E \to I^k \\ \text{with } \partial E = \partial(D^d \times I^k) \end{array} \right\} \big/ \cong \arrow[rr, "(*)"] & & \left\{ \begin{array}{l} \bar{C}_n(S^d,\infty)\text{-bundles (with str. group } Diff_\partial(D^d)) \\ \text{over } I^k \text{, trivial over } \partial I^k \end{array} \right\} \big/ \cong \end{tikzcd}$$

$$\begin{array}{ccc} E & & E\bar{C}_n(\pi) \\ \pi \downarrow & \longmapsto & \bar{C}_n(\pi) \big\downarrow \\ I^k & & I^k \end{array}$$

Schematic picture of $E\bar{C}_n(\pi)$:



where $\partial^{fib} E\bar{C}_n(\pi) = \partial^{fib,\infty} E\bar{C}_n(\pi) \cup \partial^{fib,fin} E\bar{C}_n(\pi)$

- is trivial
- & are non-trivial

because $\mathrm{Diff}_\partial(D^d)$ does not act trivially on $\partial\bar{C}_n(S^d,\infty)$

(only on the stratum corresponding to $A=\{1,2,\dots,n,\infty\}$)

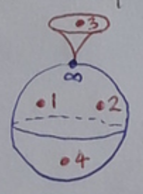
Explanations:

$$\partial\bar{C}_n(S^d,\infty) = \partial^\infty \bar{C}_n(S^d,\infty) \cup \partial^{fin} \bar{C}_n(S^d,\infty)$$

all strata corresponding to subsets $A \subseteq \{1,2,\dots,n,\infty\}$ with $\underline{\infty \in A}$

i.e. where there is a point at ∞

e.g.



$A=\{3,\infty\}$

all other strata (no point at ∞; just "collisions")

$A=\{2,4\}$

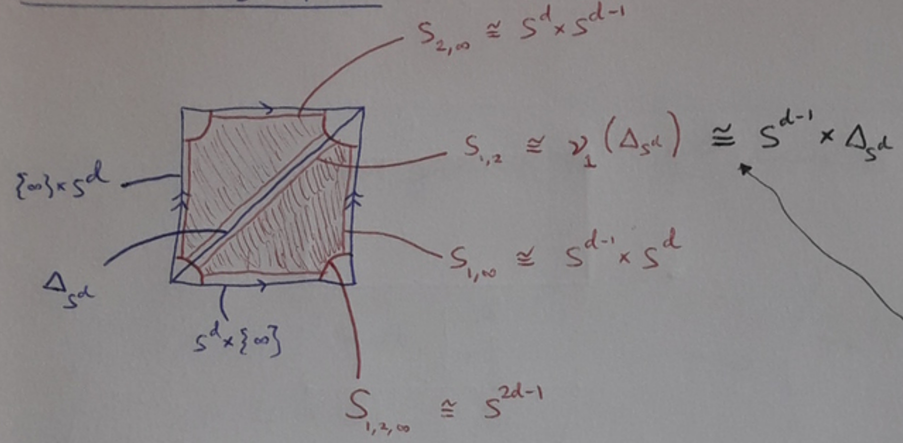
$\partial^\infty \cap \partial^{fin}$

Ψ

$\in S_{1,\infty} \cap S_{3,4}$

The action of $\mathrm{Diff}_\partial(D^d)$ preserves this decomposition.

Picture of $\bar{C}_2(S^d, \infty)$:



depends on choice of
framing of $S^d - \{\infty\}$
= trivialisation of $T(S^d - \{\infty\})$

The map $\partial \bar{C}_2(S^d, \infty) \longrightarrow S^{d-1}$:

- $$S_{1,2,\infty} - \Delta_{S^d} \cong S^{2d-1} - \Delta_{S^d} \subseteq (\mathbb{R}^d)^2 - \Delta_{\mathbb{R}^d} \xrightarrow[(x,y) \longmapsto \frac{x-y}{|x-y|}]{} S^{d-1}$$

- $S_{1,\infty} \cong S^{d-1} \times S^d \xrightarrow{\text{proj}_1} S^{d-1}$

- $S_{2,\infty} \cong S^d \times S^{d-1} \xrightarrow{\text{proj}_2} S^{d-1}$

- $\underbrace{S_{1,2} \cong S^{d-1} \times \Delta_{S^d}}_{\text{choice of framing}} \xrightarrow{\text{proj}_1} S^{d-1}$

Lemma: (1) These glue together to

canonical

$$\begin{tikzcd}
\mathbb{Z}/2 = \Sigma_2 \circlearrowright \partial^\infty \bar{C}_2(S^d, \infty) \arrow[r, "\phi_0"] \arrow[d] & S^{d-1} \circlearrowleft \mathbb{Z}/2 \text{ antipodal} \\
\Sigma_2 = \mathbb{Z}/2 \circlearrowright \partial \bar{C}_2(S^d, \infty) \arrow[ur, "\tilde{\phi}_0"'] &
\end{tikzcd}$$

choice of framing

(2) ϕ_0 and $\tilde{\phi}_0$ are $\mathbb{Z}/2$-equivariant

(3) ϕ_0 extends canonically to $\partial^{fib,\infty} E\bar{C}_2(\pi) \xrightarrow{\phi_0} S^{d-1}$

(4) given a <u>choice of trivialisation of</u> $\begin{smallmatrix} T^v E \\ \downarrow \\ E \end{smallmatrix}$ (vertical tangent bundle of $\pi \begin{smallmatrix} E \\ \downarrow \\ I^k \end{smallmatrix}$),

$\tilde{\phi}_0$ extends to $\partial^{fib} E\bar{C}_2(\pi) \xrightarrow{\tilde{\phi}_0} S^{d-1}$

"vertical framing"

(5) Admissible propagators

ref: [W, §2-3]

Note: Everything above generalises to $I^k \rightsquigarrow B$ any k-manifold with corners.

$$\left(\begin{array}{ll} \text{smooth } D^d\text{-bundle} & \begin{array}{c} E \\ \pi\downarrow \\ B \end{array} \\ \text{with } \partial E \text{ trivial} & \end{array}\right) \longrightarrow \left(\begin{array}{ll} \bar{C}_n(S^d,\infty)\text{-bundle} & \begin{array}{c} E\bar{C}_n(\pi) \\ \bar{C}_n(\pi)\downarrow \\ B \end{array}\end{array}\right)$$

All chain complexes over $\mathbb{Q}$.

Def: An admissible propagator for a smooth D^d-bundle $\begin{array}{c} E \\ \pi\downarrow \\ B^k \end{array}$ is:

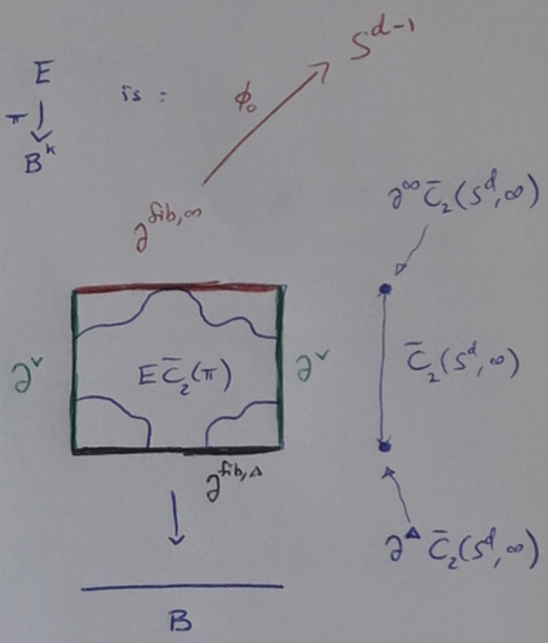
$$\theta \in Z_{k+d+1}\left(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi)\right)$$

satisfying

(i) θ transverse to $\partial^v E\bar{C}_2(\pi)$

(ii) $\partial\theta|_{\bullet}$ is $\mathbb{Z}/2$-invariant

(iii) $\partial\theta|_{\bullet} = \phi_0^{-1}(\{\pm a\})$, some $a \in S^{d-1}$



Proposition: (a) such a θ exists

(b) all such θ represent the same class in $H_{k+d+1}\left(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi)\right)$

Rmk: If $\begin{array}{c} E \\ \pi\downarrow \\ X \end{array}$ is equipped with a vertical framing, then

- $\tilde{\phi}_0 : \partial^{fib} E\bar{C}_2(\pi) \longrightarrow S^{d-1}$ is defined

- strong admissible propagator : replace (ii) + (iii) with (iv) $\partial\theta|_{\bullet\cup\bullet} = \tilde{\phi}_0^{-1}(\{\pm a\})$

- these also exist & are unique upto homology.

<u>Proof:</u> (For $B = I^k$, to simplify the argument)

- Enough to show that $\exists$ unique-up-to-homology θ satisfying (iii)

$$\left(\begin{array}{l} \text{Thom transversality} \rightsquigarrow \text{(i)} \\ c \longmapsto \tfrac{1}{2}(c + \tau.c) \rightsquigarrow \text{(ii)} \end{array} \right)$$

- Re-interpret (iii) in terms of (co)homology:

$$\begin{tikzcd}
 & {[\partial\theta]} \arrow[dl, leftrightarrow] \arrow[d, phantom, "\cap" rotate=90] & & {[\theta]} \arrow[ll, leftrightarrow] \arrow[d, phantom, "\cap" rotate=90] \\
\phi_0^*(1) \arrow[d, phantom, "\cap" rotate=90] & H_{k+d}(\partial E\bar{C}_2(\pi)) \arrow[d, phantom, "\cong" rotate=90] & & H_{k+d+1}(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi)) \arrow[ll, "\delta"'] \arrow[d, phantom, "\cong" rotate=90] \\
H^{d-1}(\partial^{fib,\infty} E\bar{C}_2(\pi)) & H^{d-1}(\partial E\bar{C}_2(\pi)) \arrow[l, "(incl)^*"] & & H^{d-1}(E\bar{C}_2(\pi)) \arrow[ll, "(incl)^*"] \arrow[lll, bend left, "(*)"] \\
1 \in H^{d-1}(S^{d-1}) = \mathbb{Q} \arrow[u, "\phi_0^*"] & & &
\end{tikzcd}$$

Need: $\exists!$ lift of $\phi_0^*(1)$ along $(*)$

$$\begin{tikzcd}
S^{d-1} & & \\
 & \partial^\infty \bar{C}_2(S^d,\infty) \arrow[ul, "\phi_0"'] \arrow[r, hook] \arrow[d, "\simeq"] & \bar{C}_2(S^d,\infty) \arrow[ull, bend right, "(\square)"'] \arrow[d, "\simeq"] \\
 & \partial^{fib,\infty} E\bar{C}_2(\pi) \arrow[uul, bend left, "\phi_0"] \arrow[r, hook] \arrow[dr] & E\bar{C}_2(\pi) \arrow[d] \\
 & & B = I^k
\end{tikzcd}
\quad \xrightarrow{H^{d-1}} \quad
\begin{tikzcd}
1 \in H^{d-1}(S^{d-1}) \arrow[d] \arrow[dd, bend right] \arrow[dr, bend left, "\cong"] & \\
H^{d-1}(-) & H^{d-1}(-) \arrow[l] \\
\phi_0^*(1) \in H^{d-1}(-) \arrow[u, "\cong"] & H^{d-1}(-) \arrow[l, "(*)"] \arrow[u, "\cong"]
\end{tikzcd}$$

<u>Claim:</u> $\exists$ map $(\square)$, which is a homotopy equivalence

proof of claim:

$$
\begin{tikzcd}
\partial^\infty \bar{C}_2(S^d,\infty) \arrow[r, "\phi_0"] \arrow[d, hook] & S^{d-1} \\
\partial \bar{C}_2(S^d,\infty) \arrow[ur, "\gamma_0"] \arrow[d, hook] & \\
\bar{C}_2(S^d,\infty) \arrow[uur, bend left, "(\square)"] \arrow[r, leftrightarrow, "\simeq"'] & C_2(S^d,\infty) \arrow[uu, "\simeq"']
\end{tikzcd}
$$

choose diffeo
$S^d - \{\infty\} \cong \mathbb{R}^d$
$\rightsquigarrow$ framing of
$T(S^d - \{\infty\})$

$(\square)$ defined on the interior by

$$C_2(S^d,\infty) \cong C_2(\mathbb{R}^d) \xrightarrow{\simeq} S^{d-1}$$

$$(x,y) \longmapsto \frac{x-y}{|x-y|}$$

$\square$.

⑥ Z_k.

Fix $\begin{array}{c} E \\ \pi \downarrow \\ B^{k(d-3)} \end{array}$ smooth D^d-bundle with ∂E trivial

Γ Choose $\theta^{(1)} \ldots \theta^{(3k)}$ adm. prop. for π

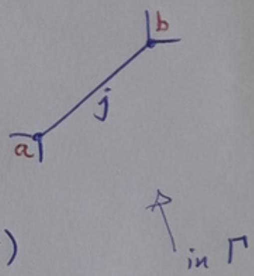
$\in Z_{k(d-3)+d+1}\left(E\bar{C}_2(\pi), \partial E\bar{C}_2(\pi)\right)$

codim $= d-1$

Γ trivalent graph with vertices $\longleftrightarrow \{1 \ldots 2k\}$
edges $\longleftrightarrow \{1 \ldots 3k\}$

$1 \leq j \leq 3k$: $\quad \phi_j : E\bar{C}_{2k}(\pi) \longrightarrow E\bar{C}_2(\pi) \circlearrowright \mathbb{Z}/2$

$\boxed{\text{forget all points except those labelled by } a \text{ and } b}$



$\longrightarrow \underbrace{\phi_j^{-1}(\theta^{(j)})}_{\mathbb{Z}/2\text{-invariant}}$ codimension $d-1$ chain on $E\bar{C}_{2k}(\pi)$

Assume that $\phi_1^{-1}(\theta^{(1)}) \ldots\ldots \phi_{3k}^{-1}(\theta^{(3k)})$ are mutually transverse

$\longrightarrow I(\Gamma) := \bigcap_{j=1}^{3k} \phi_j^{-1}(\theta^{(j)})$ codim $3k(d-1)$ chain on $E\bar{C}_{2k}(\pi)$

$\in \mathbb{Q}$.

$\longrightarrow \boxed{Z_k := \lambda_k \sum_{\Gamma} I(\Gamma).[\Gamma] \in \mathcal{A}_k}$

$\dim = k(d-3) + 2kd$
$= 3k(d-1)$