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Kontsevich characteristic classes à la Watanabe

reading seminar

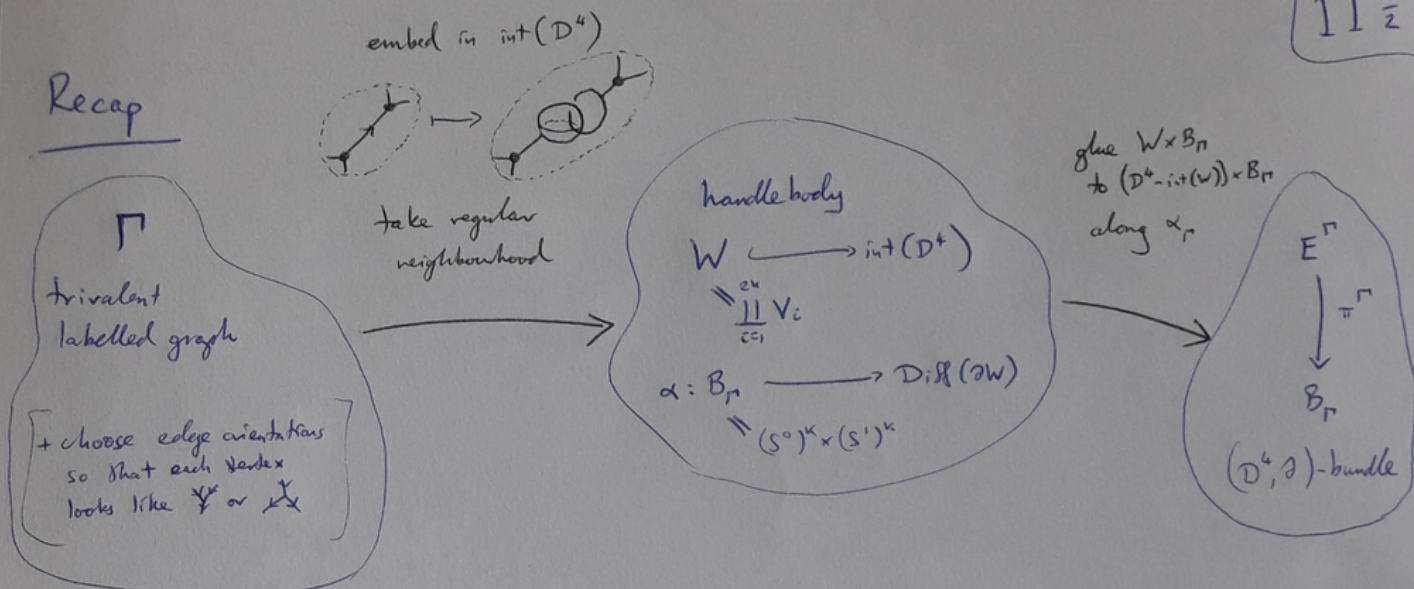
Oxford

16 June 2020

Following T. Watanabe

"Some exotic nontrivial elements of the rational
homotopy groups of $\text{Diff}(S^4)$ "

(final talk)

RecapLast week

- $\pi^\Gamma \sim$ some (D^4, ∂) -bundle over S^k
- hence $\hat{Z}_k(\pi^\Gamma)$ is defined ($\in \mathcal{A}_k$)

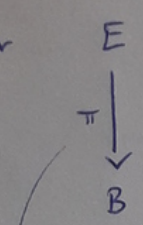
Today

- THEOREM (Watanabe) $\hat{Z}_k(\pi^\Gamma) = [\Gamma]$
- CORO (---) $\pi_k \text{BDiff}_2(D^4) \xrightarrow{\quad} \mathcal{A}_k$
- $\parallel$
- $\pi_{k-1} \text{Diff}_2(D^4)$

Plan

- Recap of §3 "graph-counting formula"
- "Key lemma" — simplification of $\uparrow$
- "Homological lemma" — conclusion of the proof.

Consider



$$(\mathbb{R}^4, \mathbb{R}^4 - \text{int}(\mathbb{D}^4))\text{-bundle} \equiv (\mathbb{D}^4, \partial)\text{-bundle}$$

Choose

$$E \xrightarrow{f} \mathbb{R}$$

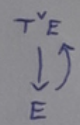
fibrewise Morse function

(restriction to each fibre is Morse)

$\cong$

v-gradient

(vertical vector field



such that ξ_b is gradient-like for f_b)

restriction to fibre over $b \in B$

$$B \xrightarrow{h} \mathbb{R}$$

Morse function

with exactly one max (index k)

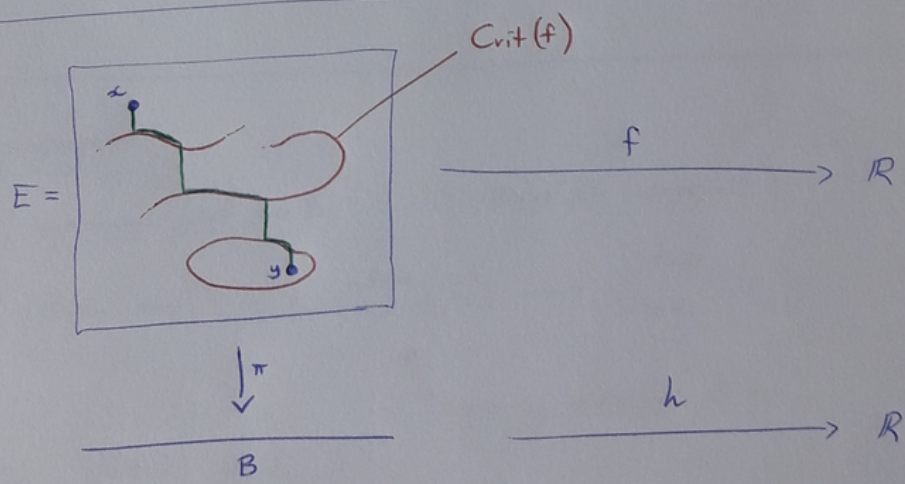
& min (index 0)

on each component of B

$\cong$

gradient-like vector field for h

Z-path : /



Finite concatenation of

- vertical
- horizontal

pieces.

contained in one fibre of π ($\Rightarrow h \circ \pi$ constant)
descending w.r.t f
flow-line of $-\xi$

contained in $\text{Crit}(f)$ ($\Rightarrow f$ constant)
descending w.r.t. $h \circ \pi$
its projection to B is a flow-line of $-\eta$

Assumptions

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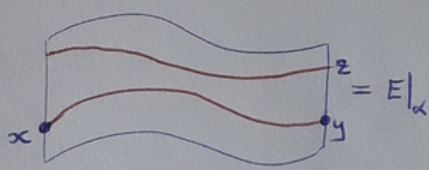
- $\forall p, q$ components of $\text{Crit}(f) \subseteq E$, $D_p \cap A_q$ is transverse

(desc. mfd)
(asc. mfd)
- $T^*D_p|_p$ is orientable
- $D_p \cap A_q = \emptyset$ if $|p| < |q|$.
- $\text{CritValues}(f_b) \subseteq \mathbb{R}$ is independent of $b \in B$
- if α is a flow-line of $-\eta$ between $a, b \in B$

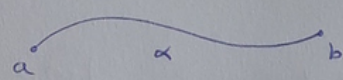
critical points with consecutive indices,

then α is transverse to the critical stratification $B = \bigsqcup_{c=0}^k B^{(c)}$ induced by f

and:



$\text{Crit}(f)$



$$\begin{aligned} \text{signed \# of } z\text{-paths } x \mapsto y \text{ in } E|_\alpha &= 1 \\ \text{signed \# of } z\text{-paths } x \mapsto z \text{ in } E|_\alpha &= 0 \end{aligned}$$

Combinatorial propagator

For each maximum point $b \in B$ ($\exists$ one per component)

$$\text{a chain map } g : C_*^{\text{Morse}}(\xi_b) \longrightarrow C_{*+1}^{\text{Morse}}(\xi_b)$$

gen. by critical pt of f_b

$$\text{with } g\partial + \partial g = \text{id.}$$

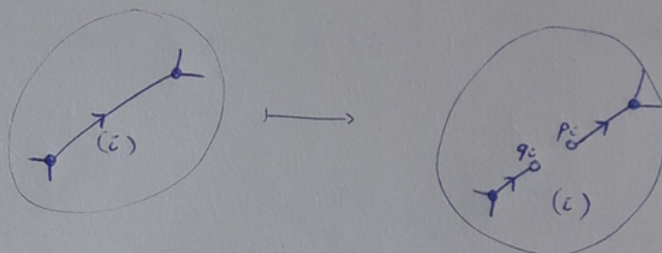
Now choose:

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- | | | |
|------------------------------|----------------------------------|-------------------------------------|
| $f^{(1)} \dots f^{(3k)}$ | piecewise Morse functions on E |] satisfying the assumptions above. |
| $\xi^{(1)} \dots \xi^{(3k)}$ | v-gradients | |
| h | Morse function on B | |
| η | gradient-like for h | |
| $g^{(1)} \dots g^{(3k)}$ | combinatorial propagation | |

C-graph

- Γ connected trivalent labelled graph $\begin{pmatrix} 3k \text{ edges} \\ 2k \text{ vertices} \end{pmatrix}$
- orientation of each edge
- for some of its edges:



$$p_i, q_i \in \text{Crit}(f) \cap \pi^{-1}(\max(h))$$

$$(\deg_i(\Gamma) = |p_i| - |q_i|)$$

Z-graph

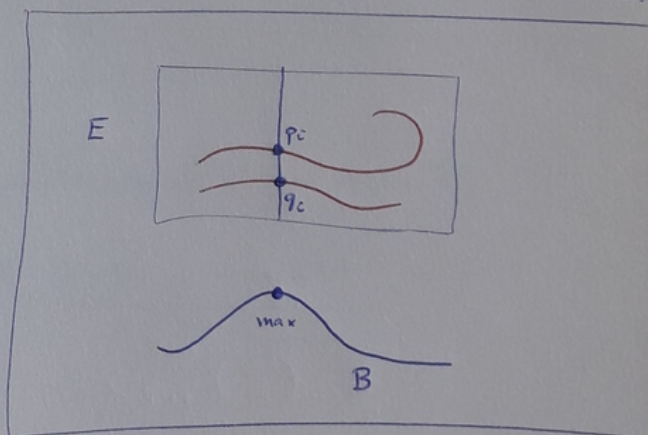
The above, plus:

- $\Gamma \hookrightarrow E$ embedding:
- univalent vertex labelled by $p \mapsto p$
- trivalent vertices $\mapsto$ same fibre of E
- non-split edge $i \mapsto$ flow-line for $-\xi^{(i)}$

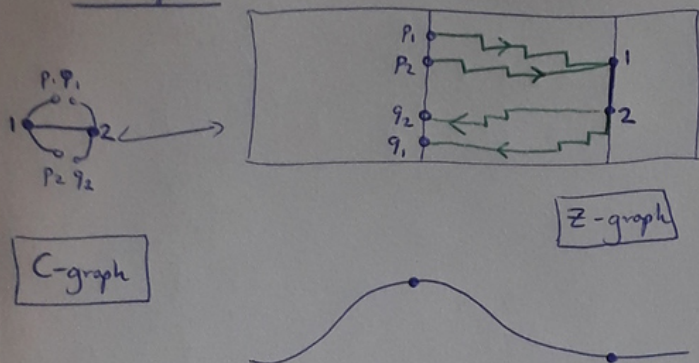
- $\begin{matrix} q_i & p_i \\ \nearrow & \searrow \\ & (i) \end{matrix} \mapsto \begin{bmatrix} \text{Z-path to } p_i \\ \text{inverse Z-path from } q_i \end{bmatrix}$ that project to the same flow-line for $-\eta$.

$$\text{wrt. } (f^{(i)}, \xi^{(i)}, h, \eta)$$

(up to piecewise smooth reparametrisation)



Example:



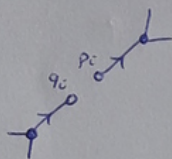
Trace of a C-graph

Γ C-graph

assume that each separated edge has degree = 1

$$\text{trace}(\Gamma) := \prod_{\substack{\text{separated edges} \\ \bar{e} \text{ of } \Gamma}} \left(-\text{coeff of } p_i \text{ in } g^{(i)}(q_i) \right) \cdot [\Gamma] \in \mathcal{A}_k$$

with its separated edges glued back together again!



$$C_*^{\text{Morse}}(\mathfrak{F}_{\max}) \xrightarrow{g^{(i)}} C_{*+1}^{\text{Morse}}(\mathfrak{F}_{\max})$$

$\downarrow \quad \quad \downarrow$
 $q_i \quad \quad p_i$

$\mathfrak{F}_{\max} = \mathfrak{F}_b$ for $b \in B$ a maximum of h .

Lemma

$$\mathcal{M}_{\Gamma}^{\mathbb{Z}} := \{ \text{all } \mathbb{Z}\text{-graphs with underlying C-graph} = \Gamma \}$$

if $f^{(i)}, \mathfrak{F}^{(i)}$ generic and each spanned edge of Γ has degree = 1

then $\mathcal{M}_{\Gamma}^{\mathbb{Z}}$ is a compact, oriented 0-dim. manifold

$$\# \mathcal{M}_{\Gamma}^{\mathbb{Z}} \in \mathbb{Z}.$$

Def

signed sum of $\mathbb{Z}$ -graphs

$$\mathbb{Z}_k^{\text{Morse}}(\pi, f^{(i)}, \mathfrak{F}^{(i)}, h, \pi, g^{(i)}) := \lambda_k \sum_{\substack{\text{C-graphs } \Gamma \text{ with} \\ \text{all separated edges} \\ \text{of degree 1}}} \# \mathcal{M}_{\Gamma}^{\mathbb{Z}} \cdot \text{trace}(\Gamma) \in \mathcal{A}_k$$

$$\lambda_k = \frac{1}{2^{3k} (2k)! (3k)!}$$

Given a choice of $f^{(-)}$ like-wise Morse functions

$\xi^{(-)}$ v-gradients

$g^{(-)} : C_*^{\text{Morse}}(\xi_{\max}^{(-)}) \longrightarrow C_{*+1}^{\text{Morse}}(\xi_{\max}^{(-)})$ combinatorial propagators

we may "turn upside-down" some of them.

$$f^{(-)} \longmapsto -f^{(-)}$$

$$\xi^{(-)} \longmapsto -\xi^{(-)}$$

$$g^{(-)} \longmapsto C_*^{\text{Morse}}(-\xi_{\max}^{(-)}) \cong C_{4-*}^{\text{Morse}}(\xi_{\max}^{(-)})^{\vee} \xrightarrow{g^{(-)\vee}} C_{3-*}^{\text{Morse}}(\xi_{\max}^{(-)})^{\vee}$$

|||
 $C_{*+1}^{\text{Morse}}(-\xi_{\max}^{(-)})$

call this $-g^{(-)}$

Def

$$\mathbb{Z}_k^{\text{av. Morse}}(\pi, f^{(-)}, \xi^{(-)}, h, \eta, g^{(-)}) := \frac{1}{2^{3k}} \sum_{(\varepsilon_1, \dots, \varepsilon_{3k}) \in \{\pm 1\}^{3k}} \mathbb{Z}_k^{\text{Morse}}(\pi, \varepsilon f^{(-)}, \varepsilon \xi^{(-)}, h, \eta, \varepsilon g^{(-)})$$

"averaged Morse"

Theorem (Graph-counting formula)

$\exists$ admissible propagators $\theta^{(1)} \dots \theta^{(3k)}$ for π such that:

$$\mathbb{Z}_k(\pi, \theta^{(-)}) = (-1)^{3k} \mathbb{Z}_k^{\text{av. Morse}}(\pi, f^{(-)}, \xi^{(-)}, h, \eta, g^{(-)})$$

"Key lemma" of §5

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For the bundles E^π constructed earlier,
 $\downarrow \pi^\pi$
 $B_\pi = (S^0)^k \times (S^1)^k$

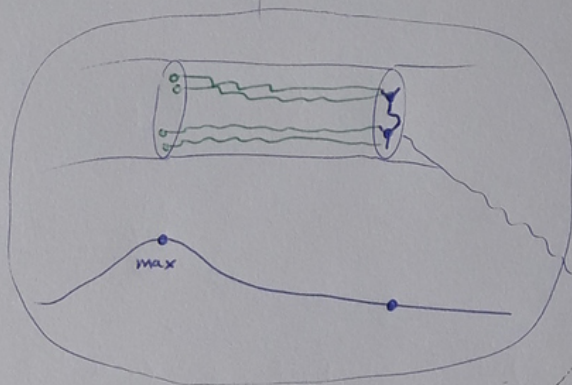
if we take (h, η) to be the "standard" Morse function & gradient on $\frac{1}{2^k} (S^1)^k$

then one may construct: likewise Morse functions $f^{(1)}, \dots, f^{(3k)} : E^\pi \rightarrow \mathbb{R}$
 - v-gradients $\xi^{(1)}, \dots, \xi^{(3k)}$
 - combinatorial propagators $g^{(1)}, \dots, g^{(3k)}$

such that

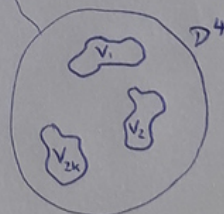
$$(a) \quad \hat{Z}_k(\pi^\pi) = (-1)^{3k} \underbrace{Z_k^{\text{av. Morse}}(\pi, f^{(1)}, \xi^{(1)}, h, \eta, g^{(1)})}_{(*)}$$

(b) a $\mathbb{Z}$ -graph $\Gamma' \hookrightarrow E^\pi$ may only contribute non-trivially to the sum $(*)$ if:



(recall that all trivalent vertices of Γ' lie in the same fibre of E^π)

each handlebody V_i contains exactly one trivalent vertex of Γ'



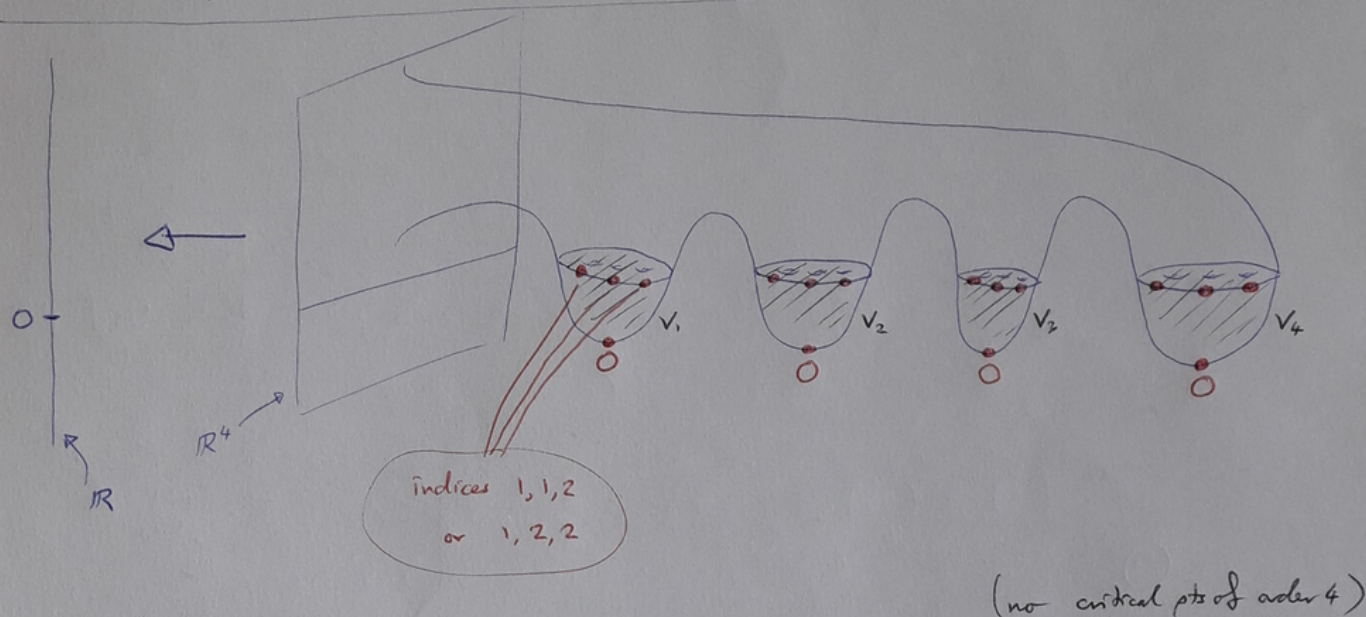
fibre of E^π containing the trivalent vertices of Γ'

in a better picture, the handlebodies V_i should be linked by Hopf links according to Γ

$$h = (S^1)^k \longrightarrow \mathbb{R}$$

$$(z_1, \dots, z_k) \longmapsto \sum_i \operatorname{Re}(z_i)$$

$$f^{(k)}|_{\text{fibre}} : \mathbb{R}^4 \longrightarrow \mathbb{R}$$



- construct bordism (of manifolds
not of bundles!) from E^n to $D^k \times S^k$

$\Rightarrow$ by Proposition last week,

$$\hat{Z}_k(\pi^n) = Z_k(\pi^n, \dots) - \alpha_k(\dots) + P_k(\dots)$$

$\uparrow$ any choice of propagators on π^n

- show that the Morse functions above (+ v-gradients) satisfy the assumptions of §3

$\Rightarrow$ by the Graph-counting formula,

$$Z_k(\pi^n, \dots) = (-1)^{3k} \cdot \overset{\text{av. Morse}}{Z}_k(\pi^n, f^{(1)}, \beta^{(1)}, h, \eta, j^{(1)})$$

$\uparrow$ some choice of propagators

- check that, with these choices,

$$\alpha_k(\dots) = 0,$$

$$\& P_k(\dots) = 0.$$

Lemma (actually proven a bit later)

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$Z_k^{\text{Morse}}(\pi, f^{(1)}, z^{(1)}, h, z, g^{(1)})$ doesn't change if we multiply some of the $f^{(i)}$ by (-1) .

Upshot

$$\hat{Z}_k(\pi^\Gamma) = \frac{(-1)^{3k}}{2^{3k} (2k)! (3k)!} \sum_{\text{labelled trivalent graphs } H} \left(\begin{array}{c} \# \text{ of } Z\text{-graphs with} \\ \text{underlying } H \end{array} \right) \cdot [H]$$

counted with sign
& with a coefficient $\in \mathbb{Q}$
depending on the combinatorial
propagation $g^{(1)} \dots g^{(3k)}$

$\Rightarrow$ Our goal is:

$$\left(\begin{array}{c} \text{signed } \# \text{ of } Z\text{-graphs} \\ \text{with underlying} \\ \text{trivalent graph } \cong H \end{array} \right) = \begin{cases} 0 & H \neq \Gamma \\ \text{non-0} & H = \Gamma \end{cases}$$

labelled trivalent graph

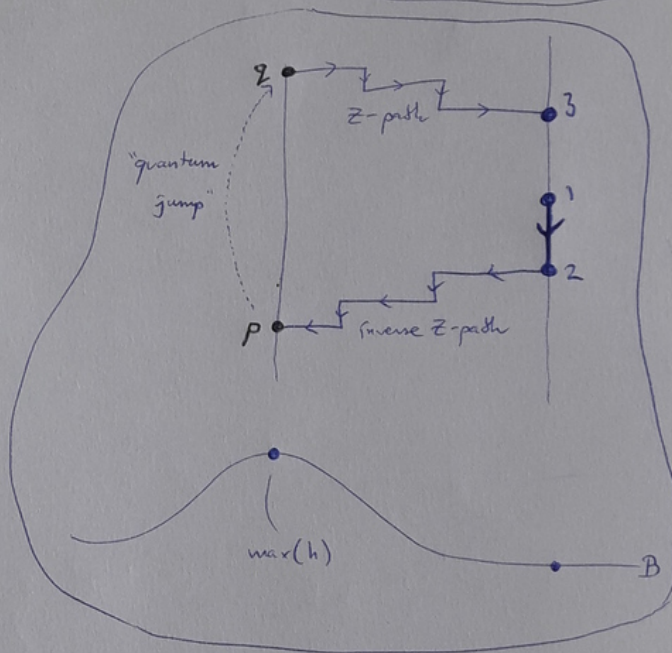
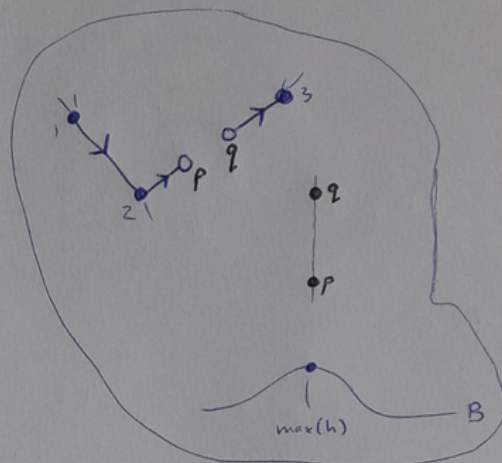
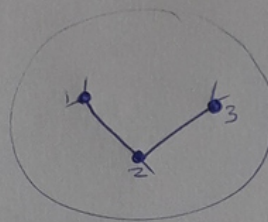
edge orientations (any!)
split some edges
label with critical points of $E^n|_{\max(h)}$

C-graph

embed into E^n
edges $\mapsto$ (inverse) $\mathbb{Z}$ -paths

$\mathbb{Z}$ -graph

$$\mathbb{I}: H \hookrightarrow E^n$$



Observation

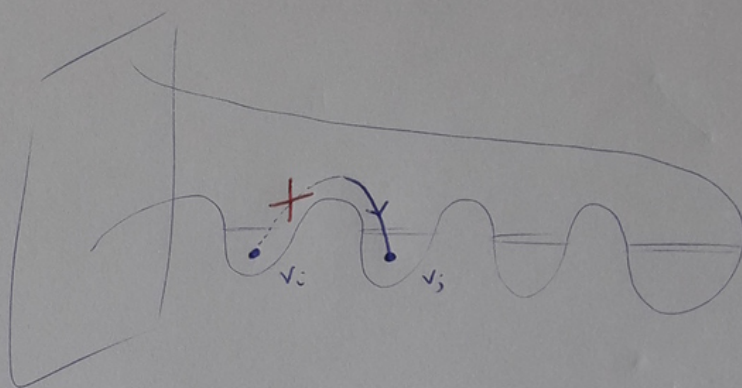
Suppose H is a C-graph counting non-trivially towards the sum $\hat{Z}_n(\pi^n)$

- Then
- all edges of H must be split
 - all outgoing univalent vertices are labelled by point of index 0, 1, 2
 - all incoming univalent vertices are labelled by point of index 1, 2, 3.

proof by picture

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(i)



non-split edge $\rightarrow$ vertical flow line in $\mathcal{S}b_{\text{loc}} \cong \mathbb{R}^4$
 between points in distinct v_i, v_j ($i \neq j$)
 (by "kg lemma" (b))

This can't exist.

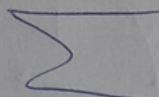
(ii) The only critical points of $f^{(i)}$ in $\mathcal{S}b_{\text{loc}}$ are $\coprod_{i=1}^{2k} v_i$ have index 0, 1, 2.

(iii) - no c. points of index 4

- $\mathbb{Z}$ -paths must have a non-trivial vertical segment $\Rightarrow$ index cannot be 0.

$\rightarrow$ combinatorial restriction on possible C-graphs H in the sum.

$$\hat{Z}_k(\pi^n) = \frac{(-1)^{3k}}{2^{3k} (2k)! (3k)!}$$


labelled
trivalent
graphs H

$$\sigma \in \Sigma_{2k}$$

(relabeling the
vertices)

$$\prod_{\text{edges } (e,e') \text{ of } H} \sum_{a,b=1}^3 \ell_k(\beta_a^{(e)}, \beta_b^{(e')}) \quad [H]$$

(*)

where $\beta_1^{(i)}, \beta_2^{(i)}, \beta_3^{(i)} =$ cores of handles of
 $V_i \subseteq \text{int}(D^4)$

comes from Γ

Obs

$$(*) \neq 0 \text{ iff } \exists \text{ isomorphism } H \cong \pm \Gamma$$

(preserving vertex-
labellings)

Hence $\hat{Z}_k(\pi^n) = \text{multiple of } [\Gamma]$

Computation (with signs!) $\leadsto$ exactly $[\Gamma]$.

□