Reidemeister moves for triple-crossing link diagrams

joint work with
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Knots and representation theory
Seminar
Moscow

Plan

- Reidemeister moves for classical (2-) diagrams
- n-diagrams $(n \geq 2)$
- Main theorem : analogue of Reidemeister moves for 3-diagrams
- sketch proof of completeness of the moves

Classical knot / link diagrams

- immersed 1-manifold in $\mathbb{R}^2$ (closed)
- embedded except at finitely many double-points, where 2 strands intersect transversely :

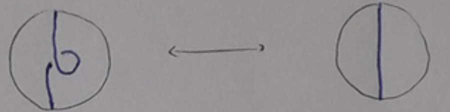
- plus "over-under" information at each double-point :

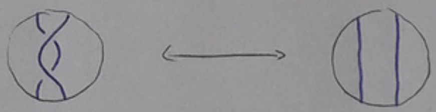
Every link has such a diagram : - project $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$
- perturb $\rightsquigarrow$ immersion
self-intersections are transverse

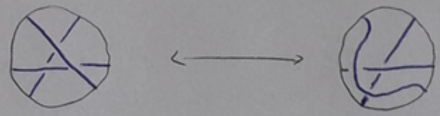
Theorem (Alexander-Briggs, Reidemeister 1927)

If two diagrams represent the same link, then they differ by :

- ambient isotopy in $\mathbb{R}^2$
- the 3 Reidemeister moves

(1) 

(2) 

(3) 

Extremely useful:

$$\left.\begin{array}{l}\text{invariant of (isotopy classes of) diagrams} \\ \quad + \text{invariant under (1), (2), (3)}\end{array}\right] \Rightarrow \text{invariant of links in } \mathbb{R}^3$$

Our main theorem is an analogue of Reidemeister moves for "3-diagrams"......

<u>n-diagrams ($n \geq 2$)</u>

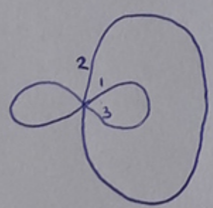
- immersed 1-manifold in $\mathbb{R}^2$ (closed)
- embedded except at finitely many <u>n-tuple points</u>, where n strands intersect transversely:

($n=3$)

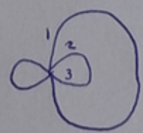
- plus "over-under" information:

Introduced by Colin Adams in 2013.

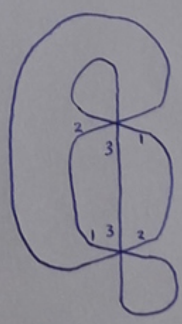
<u>Examples</u>



<u>Hopf link</u>



<u>unlink</u>



<u>trefoil</u>

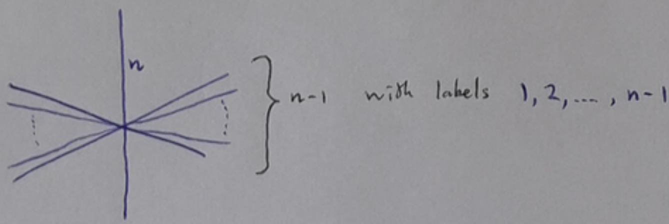
Proposition (Adams 2013)

Every link in $\mathbb{R}^3$ has an <u>n-crossing diagram</u> for all $n \geq 2$.
(n-diagram)

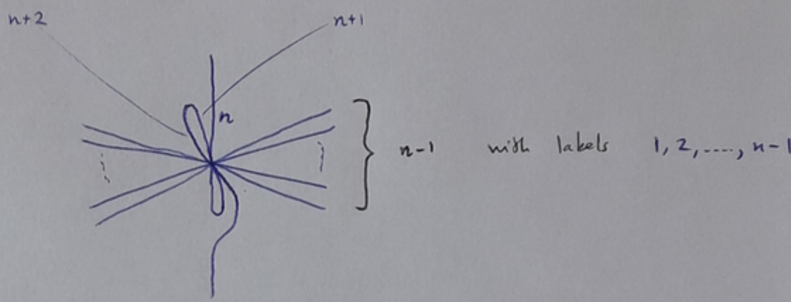
<u>proof</u>: (induction)

$n = 2$: ✓

$n \longmapsto n+2$:

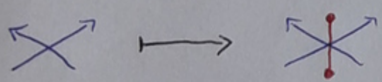


$\downarrow$



$n = 3$: <u>Fact</u> : Every 2-diagram admits a <u>CCC (crossing circle cover)</u>

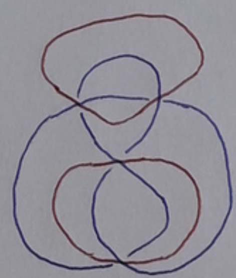
<u>proof</u>

- choose orientation of the 2-diagram
- 
- note that each complementary region contains an <u>even</u> <u>number</u> of 
- connect these in pairs by disjoint arcs (<u>non-unique</u>)

<u>Def.</u>

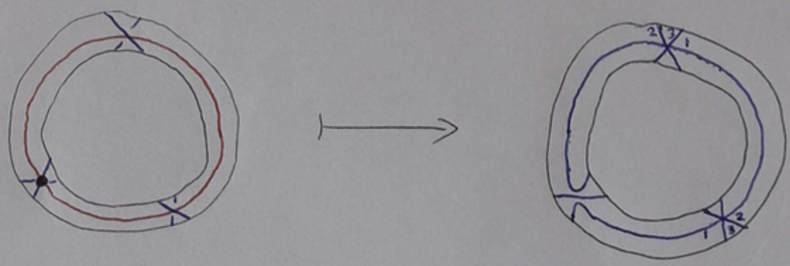
- disjoint union of embedded circles $\subseteq \mathbb{R}^2$
- intersect the diagram <u>only</u> at crossing points
- each crossing point of the diagram intersects the CCC

<u>Examples</u>



To turn a 2-diagram into a 3-diagram:

- choose a CCC + basepoint of each CC
- modify the 2-diagram as follows:



$\square$.

<u>Side note on n-crossing numbers</u>

$$c_n(L) := \min\{\#\text{ of crossings of } D \mid D \text{ is an } n\text{-diagram of } L\}$$

The proof above implies:

$c_3(L) < c_2(L)$ [L not an unlink]

$c_{n+2}(L) \leqslant c_n(L)$

Other known facts:

$c_4(L) < c_2(L)$ [L not an unlink]
$c_{kn}(L) \leqslant c_n(L)$ (k integer $\geqslant 2$)
} [Adams et al.]

$c_5(L) < c_3(L)$ [L non-split, $L \neq$ unknot, $L \neq$ Hopf] } [Adams - Hoste - P.]

Open question:

$c_{n+1}(L) \leqslant c_n(L)$ for all n?

Relations to other invariants :

- Hyperbolic volume of hyperbolic links L :

$$\boxed{\mathrm{hypvol}(L) \leqslant 4.\underbrace{v_{tetra}}_{=1.0149\ldots} + (2c_3(L)-4).\underbrace{v_{octa}}_{=3.6638\ldots}}$$ [Adams 2013]

Often better than the upper bound in terms of $c_2(L)$

$$\mathrm{hypvol}(L) \leqslant 4.v_{tetra} + (c_2(L)-5).v_{octa} \qquad (\text{if } c_2(L) \geqslant 5)$$

- Span of the Kauffman bracket polynomial $\langle L \rangle$:

$$\boxed{\mathrm{span}\langle L\rangle \leqslant \left(\left\lfloor \frac{n^2}{2}\right\rfloor + 4n - 8\right).c_n(L)} \qquad \text{for } n \geqslant 3$$ [Adams, Capovilla-Searle, Freeman, Irvine, Petti, Vitek, Weber, Zhang 2014]

Classically we have :

$$\mathrm{span}\langle L\rangle \leqslant 4.c_2(L)$$ [Kauffman; Murasugi; Thistlethwaite 1987]

- Genus :

$$\boxed{g(L) \leqslant \tfrac{1}{2}\left(c_3(L) - \#\text{components of } L + 1\right)}$$ [Jabłonowski 2020]

Classically : $g(L) \leqslant \frac{1}{2}\left(c_2(L) - \#\text{components of } L + 1\right)$

More generally : for $n \geqslant 3$ odd,

$$g(L) \leqslant \frac{1}{2}\left(\frac{(n-1)^2}{4}c_n(L) - \#\text{components of } L + 1\right)$$

Coming back to Reidemeister moves....

By the proposition above, we know that $(\forall n \geq 2)$

$$\{n\text{-diagrams}\}/_{\text{isotopy}} \longrightarrow\!\!\!\!\rightarrow \{\text{links}\}/_{\text{isotopy}}$$

is <u>surjective</u>.

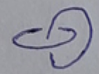
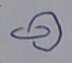
<u>Q</u>: What is its "kernel"?
Can it be described by finitely many <u>local moves</u>?

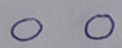
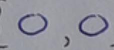
(n=2) Reidemeister moves
(n=3) ... in 5 minutes — first, a subtlety about orientations.

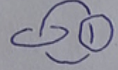
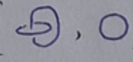
Relative orientations of links

any link L $\longrightarrow$ partition into <u>maximal non-split sublinks</u>

<u>Ex.</u>: K knot $\longrightarrow$ $\{K\}$

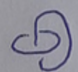
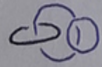
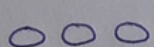
 $\longrightarrow$ {  }

 $\longrightarrow$ {  }

 $\longrightarrow$ {  }

<u>Def.</u>: A <u>relative orientation</u> of L is an orientation of each component of L, modulo orientation reversal of each maximal non-split sublink (independently).

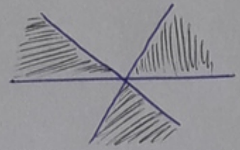
<u>Ex.</u>:

	# of rel. orientations
K knot	1
	2
	2
	1

Lemma: For n odd, each n-crossing diagram D of L <u>determines an orientation of L</u> (& hence a relative orientation)

proof: Choose a chessboard colouring of D

$\rightsquigarrow$ locally:

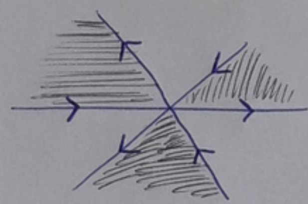


$(n=3)$

To resolve ambiguity, say that the unbounded region should be white.

Orient each strand of D so that it points <u>anticlockwise</u> around each black region.

This is compatible since n is odd:



$\square$.

Lemma: For n odd,

$$\begin{tikzcd} \{n\text{-diagrams}\}/_{\text{iso}} \arrow[r, two heads] & \{\text{oriented links}\}/_{\text{iso}} \arrow[d, two heads] \\ & \{\text{rel. oriented links}\}/_{\text{iso}} \end{tikzcd}$$

is <u>surjective</u>.

proof —— slight refinement of the proof earlier.

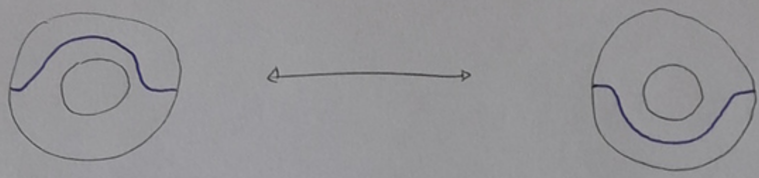
Theorem (Adams-Hoste-P. 2019) $\exists$ five moves on 3-diagrams such that:

$$\begin{tikzcd} \{3\text{-diagrams}\}/_{\text{iso}} \arrow[r, two heads] \arrow[d, two heads] & \{\text{rel. oriented links}\}/_{\text{iso}} \\ \{3\text{-diagrams}\}/_{\text{iso \& five moves}} \arrow[ur, "\text{bijection}"] & \end{tikzcd}$$

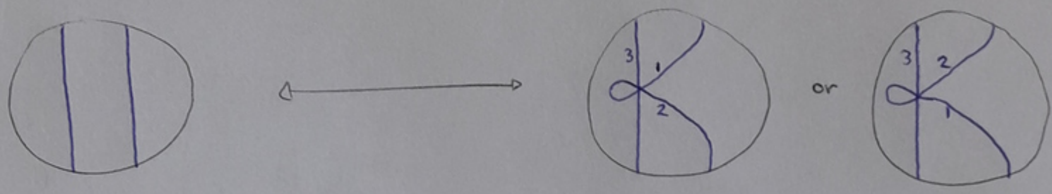
i.e. (A) each of the moves <u>preserves</u> the represented (link + rel. or.)
(B) if two 3-diagrams represent the same (link + rel. or.), then they differ by isotopy + a finite sequence of these moves.

The moves :

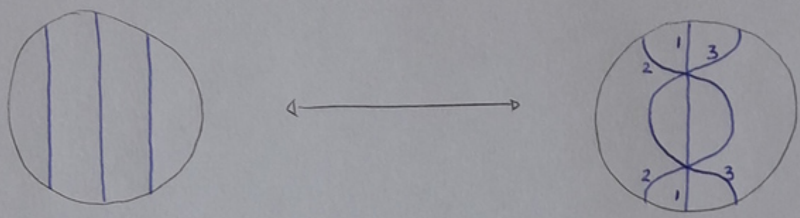
(0)



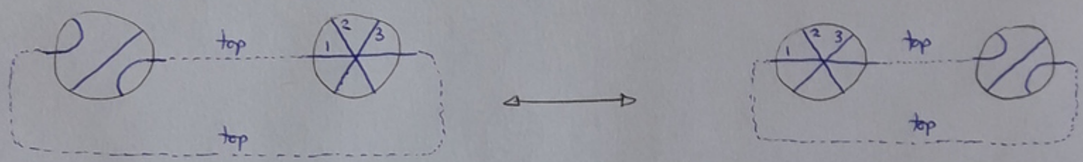
(1)



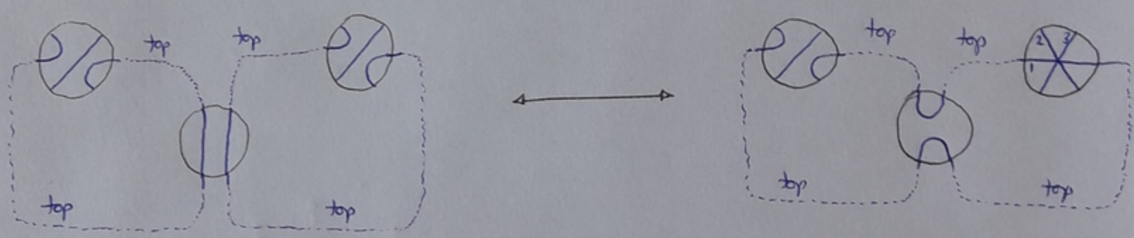
(2)



(B_p)
"basepoint"



(Band)



Note: - Each move preserves the isotopy type of the induced link.

- Moves (1), (2), (B_p), (Band):

  LHS and RHS admit chessboard colourings that are <u>equal on the boundary</u>

  (Ex: )

  => these moves preserve also the induced orientation.

- Move (0):

  <u>Reverses</u> chessboard colouring (& hence orientation) of the subdiagram surrounded by the annulus,

  But this is a union of <u>split-summands</u> of the link,

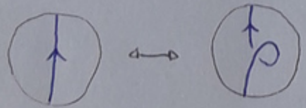
  => preserves the induced <u>relative orientation</u>.

=> proves statement (A).

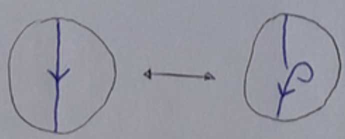
<u>Sketch proof of statement (B) — why do these moves suffice?</u>

<u>Recall</u>: $\{2\text{-diagrams}\} \big/ {\substack{\text{iso} \\ (R1),(R2)(R3)}} \xrightarrow{\cong} \{\text{links}\} \big/ \text{iso}$

[Polyak 2010]: $\{\text{oriented } 2\text{-diagrams}\} \big/ {\substack{\text{iso} \\ (\Omega 1a),(\Omega 1b), \\ (\Omega 2),(\Omega 3)}} \xrightarrow{\cong} \{\text{oriented links}\} \big/ \text{iso}$

$(\Omega 1a)$ 

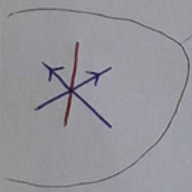
$(\Omega 2)$ 

$(\Omega 1b)$ 

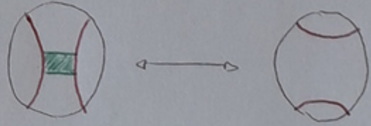
$(\Omega 3)$ 

Lemma:

$$\left\{\begin{array}{l}\text{oriented 2-diagrams}\\ +\ \text{compatible based ccc}\end{array}\right\} \Big/ \begin{array}{l}\text{iso}\\ \cdot\ \text{move base point}\\ \cdot\ +/-\ \text{trivial circles}\\ \cdot\ \text{band move}\end{array}\Big\}(\ast) \xrightarrow{\ \cong\ } \{\text{oriented 2-diagrams}\}\big/_{\text{iso}}$$



1-surgery on the ccc:



Corollary (1):

$$\left\{\begin{array}{l}\text{oriented 2-diagrams}\\ +\ \text{compatible based ccc}\end{array}\right\} \Big/ \begin{array}{l}\text{iso}\\ (\ast)\\ \{(C1a),\ (C1b),\\ \ (C2),\ (C3)\\ \text{or. reversal of conn. components}\end{array}\Big\}(\ast\ast) \xrightarrow{\ \cong\ } \{\text{rel. oriented links}\}\big/_{\text{iso}}$$

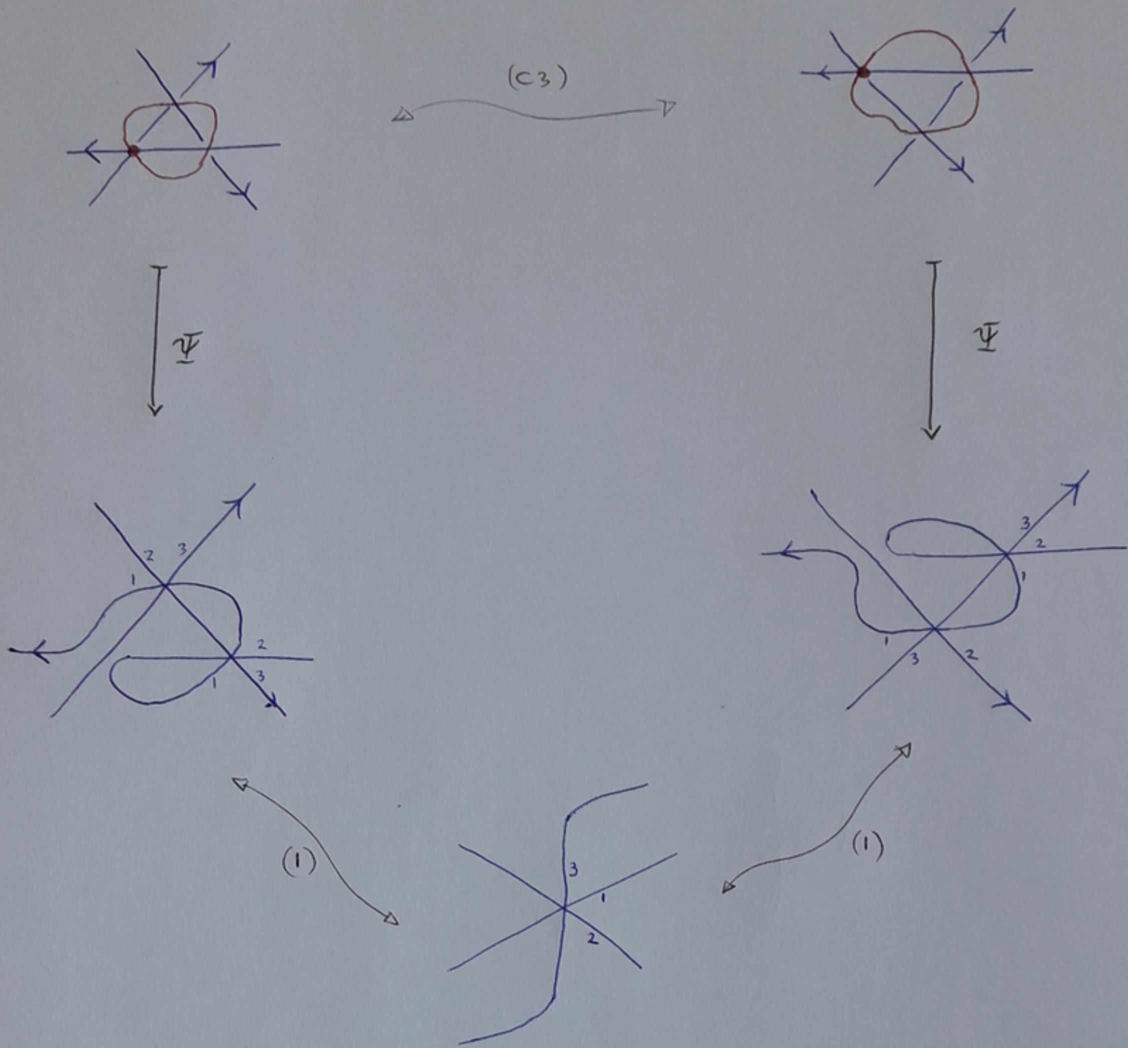
lifts of $(\Omega 1a)$, $(\Omega 1b)$, $(\Omega 2)$, $(\Omega 3)$
to oriented 2-diagrams
equipped with compatible based ccc

Proposition (2):

Denote by Ψ the operation $\left\{\begin{array}{l}\text{oriented 2-diagrams}\\ +\ \text{comp. based ccc}\end{array}\right\} \longrightarrow \{\text{3-diagrams}\}$

If $d \longleftrightarrow d'$ by $(\ast\ast)$,
then $\Psi(d) \longleftrightarrow \Psi(d')$ by $(0), (1), (2), (Bp), (Band)$.

proof for move (C3) :



Lemma(3):

$\forall$ 3-diagram D

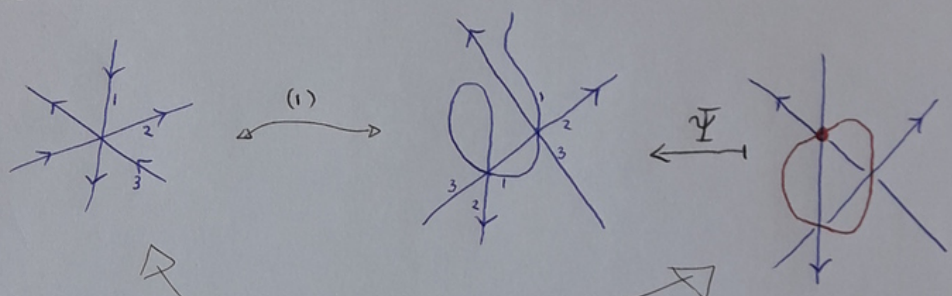
$\exists$ oriented 2-diagram + compatible based ccc $\quad d$

such that $\quad \Psi(d) \xrightarrow{(1)} D$

> " $\Psi : \left\{ \begin{array}{l} \text{or. 2-diagrams} \\ \text{+ comp. based ccc} \end{array} \right\} \longrightarrow \{\text{3-diagrams}\}$ is surjective modulo (1)-moves."

proof: Give D the canonical orientation $\rightsquigarrow$ locally 

Then:



Locally replace with to obtain d. $\square$.

End of the proof:

D_1, D_2 3-diagrams representing the same rel. oriented link.

$$\begin{tikzcd}
D_1 \arrow[r, "(1)", "\text{Lem}(3)"'] & \Psi(d_1) \arrow[rr, "{(0),(1),(2),(Bp),(Band)}"] & & \Psi(d_2) \arrow[r, "(1)", "\text{Lem}(3)"'] & D_2 \\
& d_1 \arrow[rr, "(\ast)", "\text{Coro}(1)"'] & {} \arrow[u, Rightarrow, "\text{Prop}(2)"'] & d_2 &
\end{tikzcd}$$

$\square$.