Lower central series of braid-like groups, I

joint work with

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Outline

1. LCS and the length of a group
2. Braid-like groups — our results
3. Relation to homological representations of
   - braid-like groups
   - mapping class groups
4. Disjoint-support arguments
   ↳ application to $B_n(S)$

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1 LCS and the length of a group

Def (LCS) For a group G, let
$$\Gamma_1(G) = G$$
$$\Gamma_{i+1}(G) = [\Gamma_i(G), G] \qquad i \geq 1$$

i.e. $\Gamma_i(G) = \{$ elements of the form $[\cdots[[g_1, g_2], g_3]\cdots, g_i]\}$

$$\Gamma_\infty(G) = \bigcap_{i \in \mathbb{N}} \Gamma_i(G)$$

$$\boxed{G = \Gamma_1(G) \geqslant \Gamma_2(G) \geqslant \Gamma_3(G) \geqslant \cdots\cdots \geqslant \Gamma_\infty(G)}$$

Note:

$$\begin{cases} G \text{ abelian} & \Longleftrightarrow \quad \Gamma_2(G) = 1 \\ G \ i\text{-nilpotent} & \Longleftrightarrow \quad \Gamma_{i+1}(G) = 1 \\ G \text{ residually nilpotent} & \Longleftrightarrow \quad \Gamma_\infty(G) = 1 \end{cases}$$

$$\begin{cases} 2\text{-nilpotent} & \Longleftrightarrow \quad \text{central extension of an abelian group} \\ 3\text{-nilpotent} & \Longleftrightarrow \quad \text{.................. a 2-nilpotent group} \end{cases} \quad \text{, etc.}$$

$$\begin{cases} \text{if } \Gamma_i(G) = \Gamma_{i+1}(G) \\ \text{then } \Gamma_i(G) = \Gamma_{i+1}(G) = \Gamma_{i+2}(G) = \cdots\cdots = \Gamma_\infty(G). \end{cases}$$

Def (length)

If $\boxed{\Gamma_{i-1}(G) \neq \Gamma_i(G) = \Gamma_{i+1}(G)} \longrightarrow \begin{cases} \bullet\ \Gamma_*(G) \text{ stops at } \Gamma_i \\ \bullet\ G \text{ has \underline{length} } i \\ \bullet\ \ell(G) = i \end{cases}$

Otherwise $\ell(G) = \infty$.

Note: $\ell(G) \leq i+1 \Longleftrightarrow G/\Gamma_\infty(G)$ is i-nilpotent

Refinement.

transfinite LCS $\longrightarrow$

$$\Gamma_1(G) = G$$
$$\Gamma_{\alpha+1}(G) = [\Gamma_\alpha(G), G]$$
$$\Gamma_\lambda(G) = \bigcap_{\alpha<\lambda} \Gamma_\alpha(G) \qquad \lambda \text{ limit ordinal}$$

$\ell(G)$ = smallest α such that $\Gamma_\alpha(G) = \Gamma_{\alpha+1}(G)$

but $\Gamma_\beta(G) \neq \Gamma_{\beta+1}(G) \quad \forall \beta<\alpha$.

Some known facts:

[Malcev '49] $\forall$ ordinal α $\exists$ group G with $\ell(G) = \alpha$.

Q: What about groups "in nature"?

since this is a topology seminar:
$\equiv$ "in topology"

[Cochran-Orr '97]

$D^2, S^2, T^2, \mathbb{RP}^2$, annulus, Möbius

- S compact surface $\longrightarrow$ $\begin{cases} \text{either } \pi_1(S) \text{ abelian} \\ \text{or } \boxed{\text{length}(\pi_1(S)) = \omega} \end{cases}$

and $\Gamma_\omega = 1$

[Magnus]

- $\exists$ closed 3-manifold M with $\boxed{\text{length}(\pi_1(M)) \geqslant 2\omega}$

e.g. $(K^2 \times S^1) \# L(3,1)$

[Mikhailov '16]

- For $G = \langle a, b \mid ab^{-1}a^3b = b^{-2}ab^2 \rangle \cong \pi_1(\text{closed 4-manifold})$

length$(G) = \omega^2$.

- $GL_n(\mathbb{K})$ has $\boxed{\text{length} \leq 2}$:

$$GL_n(\mathbb{K}) \geqslant \overset{\Gamma_2}{\overset{\parallel}{SL_n(\mathbb{K})}} = \overset{\Gamma_3}{\overset{\parallel}{SL_n(\mathbb{K})}} = \cdots\cdots$$

$$\left[GL_2(\mathbb{F}_2) = \Sigma_3 \geqslant C_3 = C_3 = \cdots\cdots \right]$$

<u>key fact</u> : $SL_n(\mathbb{K})$ is generated by elementary matrices

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- F_n has $\boxed{\text{length} = \omega}$ (and $\Gamma_\omega = 1$) [Magnus '35]

- B_n has $\boxed{\text{length} = 2}$ (i.e. $\Gamma_2 = \Gamma_3$) [Gorin - Lin '69]
  $[n \geqslant 2]$
  (braids of "writhe" = 0)

- $\mathrm{Mod}(S_g)$ has $\boxed{\text{length} = 1}$ (i.e. $\Gamma_1 = \Gamma_2$ "perfect") [Powell '78, building on work of Birman '70]
  $[g \geqslant 3]$

- <u>Torelli group</u> : $(g \geqslant 3)$

$$1 \longrightarrow \mathcal{T}_g \longrightarrow \mathrm{Mod}(S_g) \longrightarrow\!\!\!\!\rightarrow Sp_{2g}(\mathbb{Z}) \longrightarrow 1$$

residually nilpotent (i.e. $\Gamma_\omega = 1$) [Bass - Lubotzky '94]

- <u>analogously</u> : $(n \geqslant 3)$

$$1 \longrightarrow IA_n \longrightarrow \mathrm{Aut}(F_n) \longrightarrow\!\!\!\!\rightarrow GL_n(\mathbb{Z}) \longrightarrow 1$$

$\Gamma_\omega = \Gamma_3 = \Gamma_2 = \mathrm{Aut}^+(F_n) \underset{(2)}{\trianglelefteq} \mathrm{Aut}(F_n)$

residually nilpotent ($\Gamma_\omega = 1$) [follows since F_n is res. nilp.]

(2) Braid-like groups

S surface $\longrightarrow$ $B_n(S) = \pi_1(C_n(S))$

unordered configuration space of n points on S

$(\text{e.g. } B_n(D^2) \cong B_n)$

Thm (Darné – P – Soulié '21)

If $n \geq 3$ then $\text{length}(B_n(S)) \leq 3$ $\begin{cases} 2 & \text{non-or.} / \subseteq S^2 \\ 3 & \text{orientable \& genus} \geq 1 \end{cases}$

If $n = 2$ then $\text{length}(B_2(S)) \geq \omega$ if $S \notin \{S^2, D^2, \mathbb{RP}^2\}$

$\text{length}(B_2(S)) = 2$ if $S \in \{S^2, D^2\}$

$B_2(\mathbb{RP}^2) \cong$ dicyclic group of order 16

$\cong$ subgroup of $\mathbb{H}^*$ generated by $e^{\pi i/4}$ and j.

3-nilpotent [van Buskirk '66]

(ℓ, m, n) integers $\geqslant 0$

$\longmapsto \quad wB(\ell, m, n) = \pi_1 \left(\text{space of } \begin{bmatrix} \ell & \text{points} \\ m & \text{oriented circles} \\ n & \text{unoriented circles} \end{bmatrix} \text{ in } \mathbb{R}^3 \right)$

disjoint
unknotted, unlinked.

Note:

$wB(\ell,0,0) = \Sigma_\ell$

$wB(0,m,0) = m^{th}$ welded braid group $\longleftrightarrow$ welded knot theory.

$wB(0,0,n) = n^{th}$ extended welded braid group

$$\begin{array}{c} wB(0,0,n) \\ \cong \\ \Sigma\mathrm{Aut}(F_n) \subseteq \mathrm{Aut}(F_n) \\ \twoheaduparrow \\ \mathrm{Mod}\left(\quad ; \partial D^3 \right) \end{array}$$

Thm (Damiani - P. - Soulié)

$\begin{bmatrix} \ell \notin \{1,2\} \\ m \notin \{1,2,3\} \\ n \notin \{1,2,3\} \end{bmatrix} \longrightarrow \text{length}\left(wB(\ell,m,n)\right) = 2$

$\left[m \text{ or } n \in \{2,3\} \right] \longrightarrow \text{length}\left(wB(\ell,m,n)\right) \geqslant \omega$

$\left[\ell = 2 \text{ and } m \neq 0 \right] \longrightarrow \text{length}\left(wB(\ell,m,n)\right) \geqslant \omega$

$\begin{bmatrix} \ell = 2 \text{ and } m = 0 \\ n \notin \{1,2,3\} \end{bmatrix} \longrightarrow \text{length}\left(wB(\ell,m,n)\right) \leq 3$

(remaining cases — work in progress!)

Some other results

Thm (Darné - P. - Soulié)

$$\mathrm{Mod}\left(S^n_{g,b}\right) \text{ has length } \begin{cases} 1 & g \geq 3 \\ 2 & g = 1, 2. \end{cases}$$

$$\mathrm{Mod}\left(\mathcal{N}^n_{g,b}\right) \text{ has length } \begin{cases} 2 & \binom{g \geq 5}{n=0} \text{ or } \binom{g \geq 6 \text{, even}}{n \geq 3} \\ ? & \text{o/w.} \end{cases}$$

$$\mathrm{Aut}(F_n) \text{ has length } \begin{cases} 2 & n \geq 3 \\ 3 & n = 2 \end{cases}$$

3 Relation to homological representations

$\boxed{\text{Construction scheme for representations of } B_n(S)}$ (P. - Soulié).

also more generally
- loop-braid groups
- MCGs of surfaces
- etc.

$$C_m(S \smallsetminus n) \longrightarrow C_{(m,n)}(S) \rightleftarrows C_n(S) \qquad (\partial S \neq \phi)$$

$$\begin{tikzcd}
1 \arrow[r] & B_m(S \smallsetminus n) \arrow[r] \arrow[d, two heads, "\phi"] & B_{(m,n)}(S) \arrow[r, shift left] \arrow[d, two heads, "-/\Gamma_i"] & B_n(S) \arrow[l, dashed, bend right, hook] \arrow[r] \arrow[d, two heads, "-/\Gamma_i"] & 1 \\
1 \arrow[r] & Q \arrow[r, hook] & B_{(m,n)}(S)/\Gamma_i \arrow[r, two heads] & B_n(S)/\Gamma_i \arrow[l, dashed, bend right, hook] \arrow[r] & 1
\end{tikzcd}$$

- local system on $B_m(S \smallsetminus n)$ over $\mathbb{Z}[Q]$
- twisted homology $\longrightarrow$ $\boxed{\text{representation of } B_n(S) \text{ over } \mathbb{Z}[Q]}$

Note: $\left(S = D^2, \begin{smallmatrix} i=2 \\ m=2 \end{smallmatrix}\right) \rightsquigarrow$ Lawrence - Krammer - Bigelow repr. of B_n over $\mathbb{Z}[\mathbb{Z}^2]$.

Caveat: • representation by "twisted" $\mathbb{Z}[Q]$-module automorphisms

depends on $B_n(S) \curvearrowright Q$

• when $i=2$ it is always untwisted

As $i \geqslant 2$ varies $\longrightarrow$ tower of representations

depends on the $\boxed{\text{LCS of } B_{(m,n)}(S)}$

More generally, we study the LCS of such partitioned · surface braid groups

· welded braid groups, etc.

E.g. Thm (DPS).

$B_{(2,n)}(D^2)$ has length $\geqslant \omega$

Coro (DPS + PS) $\exists$ infinite tower of rep's of B_n over $\mathbb{Z}[Q_i]$ $i \geqslant 2$

where $\boxed{i=2} \longleftrightarrow$ LKB-representation

Q_i is $(i-1)$-nilpotent.

4 Finite-length arguments

General idea

$$G = \Gamma_1(G) \geqslant \Gamma_2(G) \geqslant \Gamma_3(G) \geqslant \cdots\cdots$$

Prop $\mathcal{L}_*(G) = \bigoplus_{i=1}^{\infty} \underbrace{\Gamma_i(G)/\Gamma_{i+1}(G)}_{\mathcal{L}_i(G)}$ is a graded Lie ring.

$$\left.\begin{array}{l} \bar{g} \in \mathcal{L}_i(G) \longrightarrow g \in \Gamma_i(G) \subseteq G \\ \bar{h} \in \mathcal{L}_j(G) \longrightarrow h \in \Gamma_j(G) \subseteq G \end{array}\right] \longrightarrow [g,h] \in \Gamma_{i+j}(G)$$

$$\longrightarrow \overline{[g,h]} \in \mathcal{L}_{i+j}(G)$$

$$\overline{[g,h]} = [\bar{g},\bar{h}]$$

Note that $\mathcal{L}_1(G) = G^{ab}$.

Lemma
- $\mathcal{L}_k(G) = [\mathcal{L}_{k-1}(G), \mathcal{L}_1(G)]$
- So $\mathcal{L}_*(G)$ is generated (as a Lie ring) by G^{ab}.

Cor If G^{ab} is cyclic, then length$(G) \leqslant 2$

proof:
- G^{ab} is generated by $\bar{g}$
- $\mathcal{L}_2(G) = [G^{ab}, G^{ab}]$ is generated by $[\bar{g},\bar{g}] = \overline{[g,g]} = 0$
- $\Gamma_2(G) = \Gamma_3(G)$ □.

Examples $(B_n)^{ab} \cong \mathbb{Z}$

$Aut(F_n)^{ab} \cong \mathbb{Z}/2 \quad (n \geqslant 3) \longleftarrow$ use presentation (Nielsen '24)

...... (Armstrong - Forrest - Vogtmann '08)

$\pi_1(S^3 \setminus K)^{ab} \cong H_1(S^3 \setminus K) \cong \mathbb{Z}$ (Alexander duality).

Cor Suppose S generates G^{ab},

and $\forall$ pair $s, t \in S$

$\exists$ representatives $g, h \in G$ such that $\boxed{g \text{ \& } h \text{ commute.}}$

Then $\Gamma_2(G) = \Gamma_3(G)$.

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e.g. if $G = \mathrm{Mod}(M)$

& $\mathrm{supp}(g)$ is disjoint

from $\mathrm{supp}(h)$.

Application to $B_n(S)$ $(n \geq 2)$.

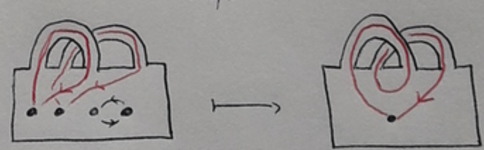
[$\mathrm{order}(\sigma) = \infty$ if $S \subseteq \mathbb{R}^2$
$\mathrm{order}(\sigma)$ finite otherwise.

Lemma (1): $\boxed{B_n(S)^{ab} \cong \pi_1(S)^{ab} \times \langle \sigma \rangle}$

More precisely: $1 \to \langle \sigma \rangle \to B_n(S)^{ab} \to \pi_1(S)^{ab} \to 1$

concatenate loops

$\sigma = \sigma_1 = \sigma_2 = \cdots = \sigma_{n-1}$



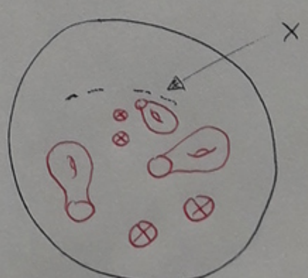
[Richards '63] S separable connected surface

$\Rightarrow$ $S \cong (S^2 \setminus X) \# (\text{countably many } \mathbb{R}P^2 \text{ or } T^2)$
along disjoint discs $\subseteq S^2 \setminus X$

+ Denjoy-Riesz
+ Schoenflies

X closed subset of Cantor set C

$X \subseteq C \subseteq [0,1] \times \{0\} \subseteq \mathbb{R}^2 \subseteq S^2$
(standard embedding)

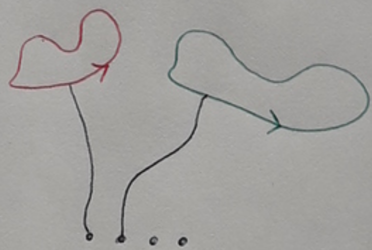


=> Lemma (2): $H_1(S)$ has a generating set where any pair may be represented by disjoint loops, except for pairs corresponding to  summands.

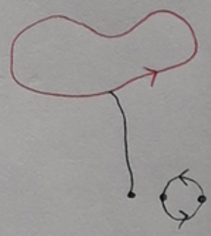
Coro: if $S \subseteq S^2$ or S is non-orientable, then $\boxed{\mathrm{length}(B_n(S)) = 2}$ for $n \geqslant 3$.

proof: • We have no  summands.

• Lemma(2) $\longrightarrow$ any pair of gen. of $\pi_1(S)^{ab} \subseteq B_n(S)^{ab}$ may be realised disjointly => commute.



• Lemma (1) $\longrightarrow$ $B_n(S)^{ab} \cong \underset{\substack{\rotatebox{90}{$\in$} \\ \gamma}}{\pi_1(S)^{ab}} \times \langle \sigma \rangle$

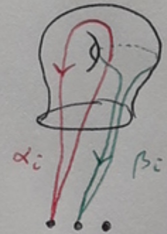


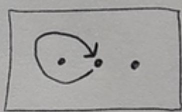
□

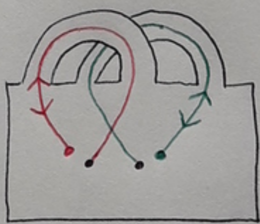
Prop. In general, $\boxed{\text{length}(B_n(S)) \leq 3}$ for $n \geqslant 3$.

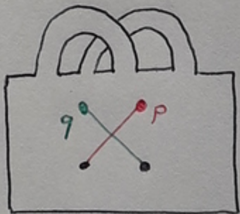
proof: Need to show that $\Gamma_3(B_n(S)) = 0$.

Above argument $\Rightarrow$ $\Gamma_2(B_n(S))$ is generated by $\overline{[\alpha_i, \beta_i]}$

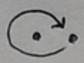
where 

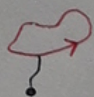
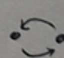
Claim: $[\alpha_i, \beta_i] =$ 

proof:  $\alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}$

$\simeq$ 

$p\,q\,p^{-1}q^{-1}$ □.

Hence $\Gamma_2 = \langle$  $\rangle$

$\Gamma_1 = B_n(S)^{ab} \cong H_1(S) \times \langle \sigma \rangle = \langle$ ,  $\rangle$

$n \geqslant 3 \longrightarrow$  and  commute

 =  $^{-2}$. Hence $\Gamma_3 = [\Gamma_2, \Gamma_1] = 0$.

□

Next week:

Warm-up — Pictures of disjoint-support arguments for

$$wB(\ell, m, n)$$
$$\mathrm{Mod}(S_{g,b}^{n})$$
$$\mathrm{Mod}(N_{g,b}^{n})$$

Then — How to prove length$(G) \geqslant \omega$,

for $G = B_2(S)$ $\quad S \notin \{S^2, D^2, RP^2\}$

$G = B_{(2,n)}(D^2)$

etc.