Lower central series of braid-like groups, II

joint work with

- Jacques Darné
- Arthur Soulié

IMAR
Topology seminar
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Reminders

Lower central series

$$\boxed{G \geqslant \Gamma_1(G) \geqslant \Gamma_2(G) \geqslant \Gamma_3(G) \geqslant \cdots\cdots\cdots \geqslant \Gamma_\infty(G)}$$

$$\Gamma_1(G) = G$$
$$\Gamma_i(G) = [\Gamma_{i-1}(G), G] = \{\text{ elements of the form } [\cdots[[g_1,g_2],g_3]\cdots\cdots, g_i]\}$$
$$\Gamma_\infty(G) = \bigcap_{i\geqslant 1} \Gamma_i(G).$$

$$\text{length}(G) = i \underset{\text{Def.}}{\iff} \Gamma_{i-1}(G) \neq \Gamma_i(G) = \Gamma_{i+1}(G)$$
$$\iff \Gamma_{i-1}\left(G/\Gamma_\infty(G) \right) \neq 0 = \Gamma_i\left(G/\Gamma_\infty(G) \right)$$
$$\iff G/\Gamma_\infty(G) \text{ is } (i-1)\text{- nilpotent.}$$

$$\text{length}(G) = \infty \underset{\text{Def.}}{\iff} \Gamma_i(G) \neq \Gamma_{i+1}(G) \quad \forall i.$$

Lie ring

$$\mathcal{L}_*(G) = \bigoplus_{i=1}^{\infty} \mathcal{L}_i(G) \qquad \mathcal{L}_i(G) = \Gamma_i(G)/\Gamma_{i+1}(G) \qquad \left(\mathcal{L}_1(G) = G^{ab}\right)$$

$$\text{length}(G) \leqslant i \iff \mathcal{L}_i(G) = 0.$$

Basic trick

if $G^{ab} = \Gamma_1(G)$ has a generating set S such that

$\forall s,t \in S$ $\exists$ representative elements $g,h \in G$ $\left(\begin{smallmatrix}\bar{g}=s \\ \bar{h}=t\end{smallmatrix}\right)$
such that g and h commute

then $\Gamma_2(G) = 0$
$\hookrightarrow$ i.e. $\mathrm{length}(G) \leq 2$.

[& similar 2-step trick to show that $\mathrm{length}(G) \leq 3$]

Last week

Applied this to $B_n(S)$ for S surface
$\underline{\underline{n \geq 3}}$

- $\mathrm{length}(B_n(S)) \leq 3$
- $\mathrm{length}(B_n(S)) \leq 2$ if $\begin{pmatrix} S \subseteq S^2 \\ \text{or } S \text{ non-orientable}\end{pmatrix}$ & $\begin{pmatrix}\text{ideal boundary of} \\ S \text{ has} \leq 1 \text{ limit pt}\end{pmatrix}$

Plan for today

Stopping results

- welded braid groups wB_n $(n \geq 4)$
- mapping class groups of surfaces

Non-stopping results

- general trick
- $B_2(S)$
- wB_2, wB_3
- $B_{2,n}$ (<u>mixed</u> braid groups)

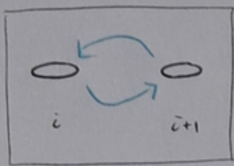
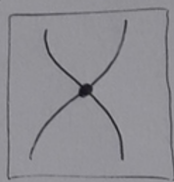
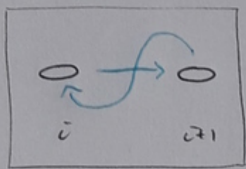
Welded braid groups

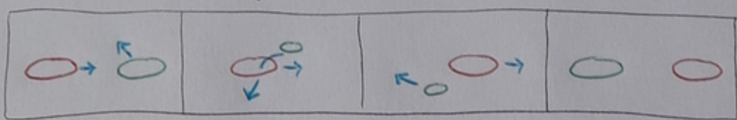
$$wB_n := \pi_1\left(\left\{\begin{array}{l}\text{oriented unlinks in } \mathbb{R}^3\\ \text{with } n \text{ components}\end{array}\right\}\right) \cong \{\text{"welded braids"}\} \leqslant \mathrm{Aut}(F_n)$$

geometric def.

diagrammatic def. $\longleftrightarrow$ welded knot theory

$\rightsquigarrow$ explicit presentation $\langle \sigma_1, \dots, \sigma_{n-1}, \tau_1, \dots, \tau_{n-1} \mid \quad \rangle$ (*)

generator	geometric meaning	diagrammatic meaning
τ_i		
σ_i		

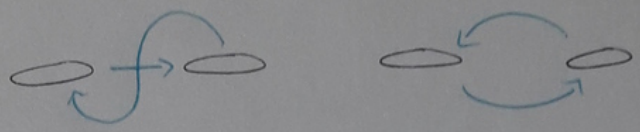


(*) Fenn - Rimányi - Rourke 1997
(via diagrammatic def.)

Brendle - Hatcher 2013
(via geometric def.)

$$\Rightarrow \quad (wB_n)^{ab} \cong \underset{\substack{\cup\\ \sigma\\ \shortparallel\\ [\sigma_i]}}{\mathbb{Z}} \oplus \underset{\substack{\cup\\ \tau\\ \shortparallel\\ [\tau_i]}}{(\mathbb{Z}/2)}$$

If $n \geqslant 4$, σ and $\tau \in (wB_n)^{ab}$ may be represented <u>disjointly</u>:



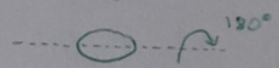
$\Rightarrow \mathcal{L}_2(wB_n) = 0$

$\Rightarrow$ lower central series of wB_n <u>stops</u> at $\Gamma_2(wB_n) = \Gamma_3(wB_n)$.

<u>NB:</u>

- We'll see that $\Gamma_*(wB_2)$ and $\Gamma_*(wB_3)$ do <u>not</u> stop.
- $\exists$ also
  - <u>virtual braid groups</u> $\boxed{vB_n}$ $\longleftrightarrow$ virtual knot theory
  - <u>extended welded braid groups</u> $\boxed{\widetilde{w}B_n}$ $\longleftrightarrow$ $\pi_1\left(\left\{\begin{matrix}\text{unoriented}\\ \text{unlinks}\end{matrix}\right\}\right)$

additional generators



and the same pattern occurs:

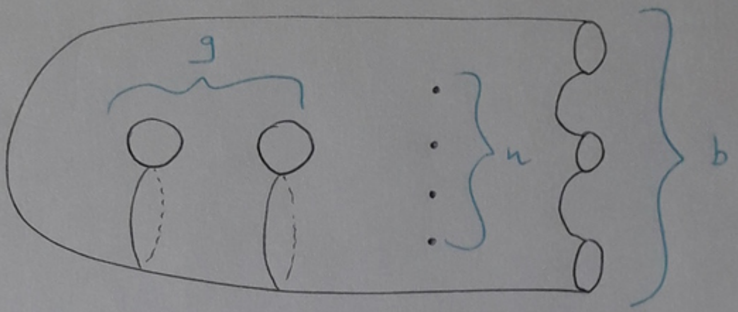
Γ_* stops at Γ_2 if $n \geqslant 4$

Γ_* does not stop if $n \in \{2,3\}$

Mapping class groups of surfaces

Orientable case

$$\Sigma_{g,b}^{n} \quad =$$



Thm [Mumford '67 (Σ_2), Powell '78 $\left(\Sigma_g,\ g \geq 3\right)$, ..., Korkmaz '03 (survey)]

Dehn twist around a non-separating simple loop

$$\mathrm{Mod}\left(\Sigma_{g,b}^{n}\right)^{ab} \cong \langle \sigma \rangle \oplus \langle T \rangle$$

where $o(\sigma) = \begin{cases} 2 & n \geq 2 \\ 1 & n \leq 1 \end{cases}$ $\qquad o(T) = \begin{cases} 1 & g \geq 3 \\ 10 & g = 2 \\ 12 & g = 1 \ \& \ b = 0 \\ \infty & g = 1 \ \& \ b \geq 1 \end{cases}$

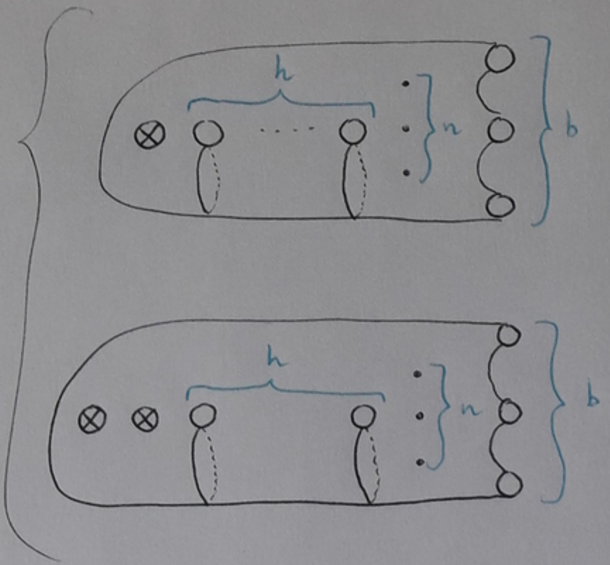
σ, T may always be represented by homeomorphisms $\Sigma_{g,b}^{n} \to \Sigma_{g,b}^{n}$ with disjoint support (=> commute)

$\Rightarrow \mathcal{L}_2\left(\mathrm{Mod}\left(\Sigma_{g,b}^{n}\right)\right) = 0$

$\Rightarrow$ lower central series of $\mathrm{Mod}\left(\Sigma_{g,b}^{n}\right)$ stops at $\Gamma_2 = \Gamma_3$.

Non-orientable case

$N^n_{g,b} =$



$g = 2h+1$

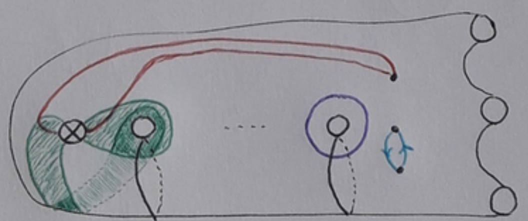
$g = 2h+2$

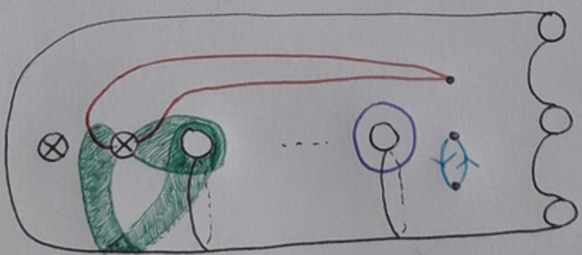
Thm [Stukow '07] Calculates $\mathrm{Mod}(N^n_{g,b})^{ab}$ for $g \geq 3$.

For $g \geq 5$ it is isomorphic to

$$\underset{y}{(\mathbb{Z}/2)} \oplus \underset{\substack{c \\ (n \geq 1)}}{(\mathbb{Z}/2)} \oplus \underset{\substack{\sigma \\ (n \geq 2)}}{(\mathbb{Z}/2)} \oplus \underset{\substack{T \\ (g \in \{5,6\})}}{(\mathbb{Z}/2)}$$

where:





"y-homomorphism" supported in a genus-2 subsurface

puncture slide

Dehn twist

puncture slide

Thm (Darné - P. - Soulié '21)

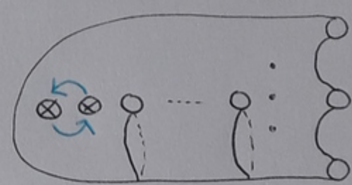
if $(g \geq 5 \ , \ n = 0)$ then the lower central series of
or $(g \geq 6 \text{ even} \ , \ n \neq 2)$ $\mathrm{Mod}(N_{g,b}^{n})$ stops at Γ_2.

proof
We need to show that these 4 generators may be represented with disjoint support (when they exist).

- $\boxed{n=0}$ ⟶ only y and T ⟶ disjoint since $g \geq 5$. ✓

- $\boxed{n \neq 2}$ ⟶ only "problem" is y and c

⟶ even genus ⟹ c may be conjugated by



to obtain c' such that · $[c'] = [c]$ (in ab.")
· $\mathrm{supp}(c') \cap \mathrm{supp}(y) = \emptyset$

□.

Other cases: $n=2$ ⟶ Conj: lower central series does not stop.

low genus ⟶ ?

high odd genus ⟶ ??

Non-stopping results

General trick

$$G \xrightarrow{\pi} H \qquad \text{surjective homomorphism}$$

Obs:

- $\pi(\Gamma_i(G)) = \Gamma_i(H)$
- if $\Gamma_*(G)$ stops at Γ_i, then $\Gamma_*(H)$ stops at Γ_i.
- if $\Gamma_*(H)$ does not stop, then neither does $\Gamma_*(G)$.

Facts: (easy to compute by hand) For $i \geq 2$:

$$\Gamma_i(\mathbb{Z} \rtimes (\mathbb{Z}/2)) = 2^{i-1}\mathbb{Z} \rtimes \{1\}$$

$$\Gamma_i(\mathbb{Z}^2 \rtimes \Sigma_2) = 2^{i-2}(\delta\mathbb{Z}) \rtimes \{1\} \qquad \delta\mathbb{Z} = \{(n,-n)\} \leqslant \mathbb{Z}^2$$

So the groups $\left(\begin{array}{l} G = \mathbb{Z} \rtimes (\mathbb{Z}/2) \\ G = \mathbb{Z}^2 \rtimes \Sigma_2 = \mathbb{Z} \wr \Sigma_2 \end{array} \right)$

have non-stopping lower central series. [In fact they have length $= \omega$ and are residually nilpotent.]

Strategy

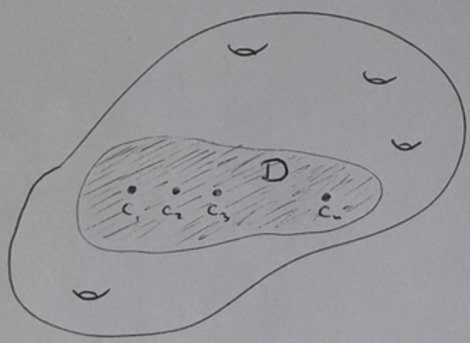
For groups of interest (today: $B_2(S)$, wB_3, wB_2, $B_{2,n}(D^2)$) try to construct a quotient onto either $\mathbb{Z} \rtimes (\mathbb{Z}/2)$ or $\mathbb{Z} \wr \Sigma_2$.

$B_2(S)$

S any surface

$$B_n(S) \xrightarrow{\ \pi\ } \pi_1(S) \wr \Sigma_n = \pi_1(S)^n \rtimes \Sigma_n \, .$$

Fix
- an ordering $(c_1, \ldots, c_n)$ of the base configuration in $C_n(S)$,
- a subdisc $D \subseteq S$ containing the base configuration.



$$\gamma \in B_n(S) = \pi_1 C_n(S) \longmapsto \pi(\gamma) := (\bar{\gamma}_1, \ldots, \bar{\gamma}_n ; \sigma)$$

γ_i = path in S starting at c_i

$\bar{\gamma}_i$ = induced <u>loop</u> in $S/D \cong S$

σ = permutation of $(c_1, \ldots, c_n)$ induced by γ

e.g.

Fact (from class$^{\text{n}}$ of surfaces)

If S is a connected surface, $S \notin \{ \mathbb{S}^2, \mathbb{R}^2, \mathbb{RP}^2 \}$,

then $\exists\ \pi_1(S) \twoheadrightarrow \mathbb{Z}$.

Coro (Darné - P. - Soulié '21)

If $S \notin \{\mathbb{S}^2, \mathbb{R}^2, \mathbb{RP}^2\}$, the lower central series of $B_2(S)$ does not stop.

proof:

$$B_2(S) \twoheadrightarrow \pi_1(S) \wr \Sigma_2 \twoheadrightarrow \mathbb{Z} \wr \Sigma_2 . \qquad \square.$$

Note:

$$\left.\begin{aligned} B_2(\mathbb{R}^2) &\cong \mathbb{Z} \\ B_2(\mathbb{S}^2) &\cong \mathbb{Z}/2 \end{aligned}\right] \longrightarrow \text{LCS stops at } \Gamma_2$$

$B_2(\mathbb{RP}^2) \cong$ dicyclic group of order 16 [van Buskirk '66]

$\longrightarrow$ LCS stops at Γ_4.

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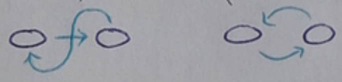
wB_2 and wB_3

presentation

$\sigma \longleftrightarrow b \rtimes (-1)$
$\tau \longleftrightarrow 1 \rtimes (-1)$

$$wB_2 \cong \mathbb{Z} * (\mathbb{Z}/2) \cong \underbrace{F_2}_{\langle a,b\rangle} \rtimes \{\pm 1\}$$



σ $\quad$ τ

$$\downarrow$$

$$\mathbb{Z}^2 \rtimes \{\pm 1\} = \mathbb{Z} \wr \Sigma_2 .$$

$$wB_3 = \langle \sigma_1, \sigma_2, \tau_1, \tau_2 \mid \ldots \rangle$$

$$\downarrow$$

$\sigma \longleftrightarrow 0 \rtimes (-1)$
$\tau \longleftrightarrow 1 \rtimes (-1)$

$$\underset{\sigma}{(\mathbb{Z}/2)} * \underset{\tau}{(\mathbb{Z}/2)} \cong \mathbb{Z} \rtimes \{\pm 1\} .$$

<u>Note:</u>

- $\Gamma_\infty(wB_2) = 1$
- but $\Gamma_\infty(wB_3) \neq 1$, and

$$wB_3 \big/ \Gamma_\infty(wB_3) \cong \left\langle \sigma, \tau \;\middle|\; \begin{matrix} \tau^2 = 1 \\ \sigma^2\tau = \tau\sigma^2 \end{matrix} \right\rangle .$$

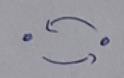
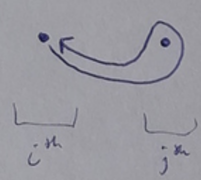
$$= wB_3 \Big/ \begin{matrix} \sigma_1 = \sigma_2 \\ \tau_1 = \tau_2 \end{matrix}$$

<u>$B_{2,n}(D^2)$</u>. The case relevant for constructing a "pro-nilpotent" tower of representations of B_n, extending the LKB (Lawrence-Krammer-Bigelow) representation.

<u>$n \geq 3$</u>.

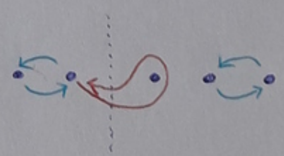
<u>Lemma</u>.

$$\left(B_{(n_1,\ldots,n_k)}\right)^{ab} = \text{free abelian on}$$

- one  for each $i : n_i \geq 2$ (i^{th} block)
- one  for each $1 \leq i < j \leq k$.

In particular,

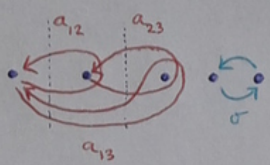
$$\left(B_{2,n}\right)^{ab} \cong \mathbb{Z}^3 \quad \text{generated by}$$



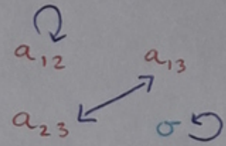
$\longrightarrow$ disjoint support argument fails!

$$\begin{tikzcd}
1 \arrow[r] & B_{1,1,n} \arrow[r, hook] \arrow[d, two heads] & B_{2,n} \arrow[r, two heads] \arrow[d, two heads] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & (B_{1,1,n})^{ab} \arrow[r, hook] \arrow[d, phantom, sloped, "\cong"] & B_{2,n}/\Gamma_2(B_{1,1,n}) \arrow[r, two heads] & \mathbb{Z}/2 \arrow[r] \arrow[d, phantom, sloped, "\ni"] & 1 \\
& \mathbb{Z}^4 & & \tau &
\end{tikzcd}$$

generated by



<u>action of τ</u> :



$$\rightsquigarrow \quad B_{2,n}/\Gamma_2(B_{1,1,n}) \cong \underset{\sigma}{\mathbb{Z}} \times \Big(\underset{\substack{a_{13} \\ a_{23}}}{\mathbb{Z}^2} \rtimes \underset{\tau}{\mathbb{Z}} \Big) \qquad (a_{12} = \tau^2)$$

$$\begin{tikzcd}
\mathbb{Z} \times (\mathbb{Z}^2 \rtimes \mathbb{Z}) \arrow[d, two heads] \\
\mathbb{Z}^2 \rtimes \mathbb{Z} \arrow[d, two heads] \\
\mathbb{Z}^2 \rtimes (\mathbb{Z}/2) = \mathbb{Z} \wr \Sigma_2 .
\end{tikzcd}$$

This proves:

<u>Thm</u> (Darné - P. - Soulié '21)

The lower central series of $B_{2,n}$ (for $n \geq 3$) does not stop.

($n=1$): Same proof works.

($n=2$): Different proof $\Rightarrow$ quotient onto $\mathbb{Z} \rtimes (\mathbb{Z}/2)$ $\Rightarrow$ LCS doesn't stop.