Signature invariants related to the unknotting number

following [Livingston] — arXiv 2017
↳ Pacific J. Math 2020

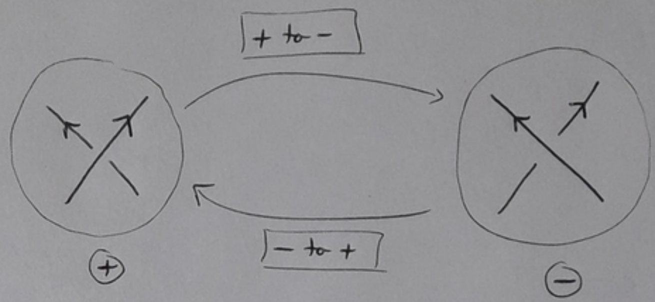
Reading seminar
Oxford
16 Feb 2021

Def K knot

$u(K)$ = min # of crossing changes to modify $K \rightsquigarrow$ unknot

$u_-(K)$ = ——— $\boxed{- \text{ to } +}$ ———

$u_+(K)$ = ——— $\boxed{+ \text{ to } -}$ ———



(indep. of choice of orientation).

$d_g(K, J)$ = min # of crossing changes to modify $K \rightsquigarrow J$

Obs $u(K) \geqslant u_-(K) + u_+(K)$

$d_g(K, J) \leqslant u(K) + u(J)$ [more generally, d_g is a metric]

$K \longrightarrow$ Levine-Tristram signature function $\sigma_K : S^1 \longrightarrow \mathbb{Z}$ (constant except at finitely many discontinuities) $\longrightarrow \max(\sigma_K) - \min(\sigma_K) \in \mathbb{N}$

Known:

$$u(K) \geqslant \underbrace{\tfrac{1}{2}\left(\max(\sigma_K) - \min(\sigma_K)\right)}_{\overset{\Vert}{u_1(K)}}$$

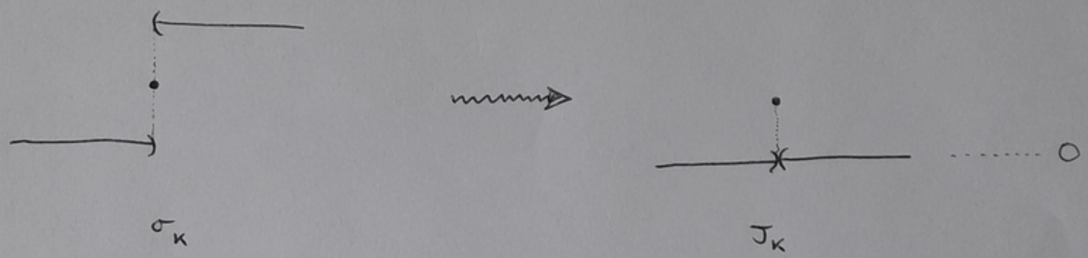
Aim: Get better lower bounds for $u(K)$ out of σ_K.

Since σ_K has <u>isolated</u> discontinuities, it makes sense to define:

$$J_K\left(e^{2\pi i t}\right) = \frac{1}{2}\left(\lim_{s\to t^+} \sigma_K\left(e^{2\pi i s}\right) - \lim_{s \to t^-} \sigma_K\left(e^{2\pi i s}\right)\right)$$

<u>"jump function" of K</u> (zero except at finitely many points)

<u>Picture:</u>



<u>Theorem.</u> K knot

$\delta \in \mathbb{Q}[t]$ irreducible

$\alpha_1, \ldots, \alpha_j$ some roots of δ on S^1

$$J := \max_i \left(|J_K(\alpha_i)|\right)$$

$$\overline{\sigma} := \max_i \left(\sigma_K(\alpha_i)\right)$$

$$\underline{\sigma} := \min_i \left(\sigma_K(\alpha_i)\right)$$

Then:

$$u_-(K) \geqslant \frac{1}{2}\left(J + \overline{\sigma}\right)$$

$$u_+(K) \geqslant \frac{1}{2}\left(J - \underline{\sigma}\right)$$

In particular,

$$\boxed{u(K) \geqslant J + \frac{1}{2}\left(\overline{\sigma} - \underline{\sigma}\right)}$$

Optimised version:

$$u_-(K) \geq \frac{1}{2} \max_{\delta, \alpha_i} \left(J + \overline{G} \right)$$

$$u_+(K) \geq \frac{1}{2} \max_{\delta, \alpha_i} \left(J - \underline{G} \right)$$

$$u(K) \geq \underbrace{\frac{1}{2} \left(\max_{\delta, \alpha_i} \left(J + \overline{G} \right) + \max_{\delta, \alpha_i} \left(J - \underline{G} \right) \right)}_{u_2(K)}$$

maximise separately!

Obs (1) $u_1(K) \leq u_2(K) \leq 2 . u_1(K)$ —— proof later

(2) maximising separately (using different polynomials δ for u_- and u_+) generally gives better lower bounds

(3) The improvement of u_2 over u_1 comes from the J factor.

So it's good to choose δ to have roots at the discontinuities of σ_K.

$\{\text{discontinuities of } \sigma_K\} \subseteq \{\text{roots of } \Delta_K\}$

So it's good to take $\delta =$ irreducible factor of Δ_K

$\{\alpha_i\} =$ all roots of δ on S^1.

Proof that $u_1(K) \leqslant u_2(K) \leqslant 2.u_1(K)$:

$2u_2 \leqslant 4u_1$

$$\max_{\delta,\alpha_i}\left(J+\overline{\sigma}\right) + \max_{\delta,\alpha_i}\left(J-\underline{\sigma}\right) \leqslant \max|J_K| + \max(\sigma_K) + \max|J_K| - \min(\sigma_K)$$

$$\boxed{\text{by definition,} \quad \max|J_K| \leqslant \tfrac{1}{2}\left(\max(\sigma_K) - \min(\sigma_K)\right)}$$

$$\leqslant 2.\left(\max(\sigma_K) - \min(\sigma_K)\right)$$

$2u_1 \leqslant 2u_2$

$\max(\sigma_K)$ is attained on an open interval in S^1

$\rightsquigarrow \quad \max(\sigma_K) = \sigma_K(\alpha_1) \qquad \alpha_1 =$ root of irred. $\delta_1 \in \mathbb{Q}[t]$

[roots of rational polynomials are <u>dense</u> in S^1 — e.g. we can always take δ_1 to be a cyclotomic poly.]

Similarly, $\quad \min(\sigma_K) = \sigma_K(\alpha_2) \qquad \alpha_2 =$ root of irred. $\delta_2 \in \mathbb{Q}[t]$

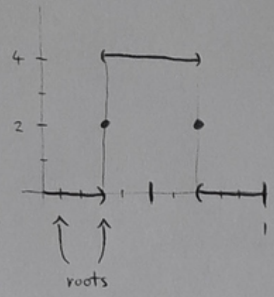
$$2.u_2(K) \geqslant \underbrace{\left(J+\overline{\sigma}\right)}_{\text{for } \delta_1, \alpha_2} + \underbrace{\left(J-\underline{\sigma}\right)}_{\text{for } \delta_2, \alpha_2} = 0 + \sigma_K(\alpha_1) + 0 - \sigma_K(\alpha_2)$$

$$= \max(\sigma_K) - \min(\sigma_K)$$

$$= 2.u_1(K).$$

Examples

$$[0,1] \xrightarrow{e^{2\pi i t}} S^1$$

K	σ_K	u_1	$\delta = \varphi_{10}$
$-(5_1 \# 10_{132})$		2	roots $= e^{2\pi i/10}, \; e^{2\pi i 3/10}$ $u_- \geqslant \frac{1}{2}(2+2) = 2$ $u_+ \geqslant \frac{1}{2}(2-0) = 1$ $\underline{u \geqslant 3}$ $\quad (u = 3)$

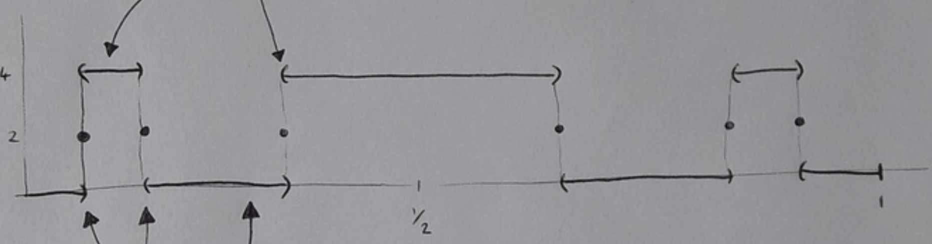
roots of $\delta_3 = t^6 - 3t^5 + 3t^4 - 3t^3 + 3t^2 - 3t + 1$

$= \Delta_{8_2}$

connected sum of

$3_1 \quad 3_1 \quad -5_1 \quad -8_2$

$10_{132} \quad -11_{n6}$



roots of $\delta_2 = t^4 - t^3 + t^2 - t + 1$

$= \Delta_{5_1}$

root of $\delta_1 = t^2 - t + 1$

$= \Delta_{3_1}$

$= \varphi_6$

$$u_1 = \tfrac{1}{2}(\max(\sigma) - \min(\sigma)) = 2$$

$$u_- \geqslant \tfrac{1}{2}(J + \overline{G}) = \tfrac{1}{2}(2+4) = 3 \qquad (\text{using } \delta_3)$$

$$u_+ \geqslant \tfrac{1}{2}(J - \underline{G}) = \tfrac{1}{2}(2-0) = 1 \qquad (\text{using } \delta_2)$$

Hence $\underline{u \geqslant 4}$.

(This is a better bound than can be obtained from any one of $\delta_1, \delta_2, \delta_3$ on their own.)

Plan:

- Some consequences of the main theorem
- Definition of σ_K (and J_K)
- Proof of theorem —— 2 steps

Gordian distance

- $u_2(K)$ depends only on σ_K, which is a concordance invariant.
- (we will see that) $u_2(K)$ actually gives a lower bound on the # of crossing changes needed to convert $K \rightsquigarrow K'$ with $\sigma_{K'} \equiv 0$.
- if $K \longrightarrow J$ with n crossing changes

  then $K\#(-J) \rightsquigarrow J\#(-J)$ —— " ——

  slice $\Rightarrow \sigma \equiv 0$

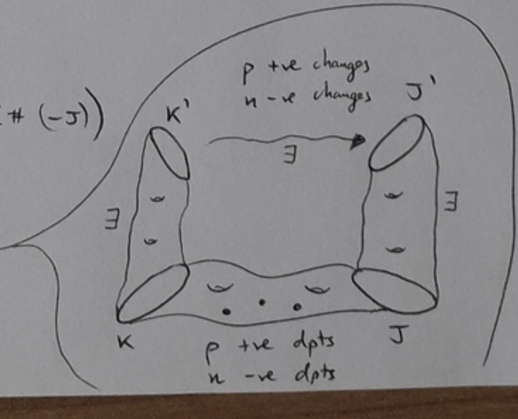
  so $u_2(K\#(-J)) \leqslant n$

Cor: $\boxed{d_g(K,J) \geqslant u_2(K\#(-J))}$

Similarly, if $d_c(K,J)$ = min # of double points in a singular concordance from K to J

then $d_c(K,J) \geqslant u_2(K\#(-J))$

uses [Owens-Strle] 2016



Def of σ_κ

$$W(\mathbb{Q}(t)) = \{\underbrace{\text{nonsingular Hermitian}}_{\substack{\text{w.r.t. involution} \\ t \mapsto t^{-1}}} \text{ matrices over } \mathbb{Q}(t)\} /_\sim$$

$$A \sim B \iff \operatorname{rk}(A) = \operatorname{rk}(B) \mod 2$$

and the Hermitian form $A \oplus (-B)$ vanishes on a subspace of dimension $\frac{1}{2}(\operatorname{rk}(A) + \operatorname{rk}(B))$

Note: $A \approx B$ (congruent) $\iff$ represent the same form w.r.t. different bases.

$$A \approx B \Downarrow A \sim B$$

$$\mathbb{Q}(t) \xrightarrow{ev_\alpha} \mathbb{C}$$

not always defined!

$$\begin{array}{c} A(t) = B(t) \in W(\mathbb{Q}(t)) \\ \Big\downarrow ev_\alpha \quad \alpha \in S^1 \text{ (almost all)} \\ A(\alpha) = B(\alpha) \in W(\mathbb{C}) \\ \Big\downarrow \sigma \\ \underbrace{\sigma(A(\alpha))}_{\text{indep. of representative.}} = \sigma(B(\alpha)) \in \mathbb{Z} \end{array}$$

$$\begin{array}{ccc} W(\mathbb{Q}(t)) & \longrightarrow & \{\text{loc. constant functions } S^1 - (\text{finite}) \longrightarrow \mathbb{Z}\} \\ & \searrow^{\sigma} & \Big\downarrow \text{"two-sided average"} \\ & & \{\text{step functions } S^1 \longrightarrow \mathbb{Z}\} \end{array}$$

$$\begin{array}{ccl} K \text{ knot} & \rightsquigarrow & \underline{\underline{\text{choose}}} \text{ Seifert surface } F \text{ \& basis of } H_1(F) \\ & & \downarrow \\ & & \text{Seifert matrix } V_F \\ & & \downarrow \\ & & W_F = (1-t)V_F + (1-t^{-1})V_F^T \in M_{2g,2g}(\mathbb{Q}(t)) \end{array}$$

<u>Fact</u> W_F is non-singular & Hermitian

<u>Thm</u> (Levine) W_F depends only on K up to Witt equivalence.

$$\begin{tikzcd} \{\text{knots}\} \arrow[r] \arrow[rr, bend right, "\sigma"'] & W(\mathbb{Q}(t)) \arrow[r, "\sigma"] & \{\text{step functions } S^1 \to \mathbb{Z}\} \end{tikzcd}$$

As before,

$$J_K(e^{2\pi i t}) = \frac{1}{2}\left(\lim_{s \to t^+} \sigma_K(e^{2\pi i s}) - \lim_{s \to t^-} \sigma_K(e^{2\pi i s}) \right).$$

Step 1.

$$\begin{tikzcd} W_F \arrow[r, "\text{change of}", "\text{basis}"'] & \text{diagonal matrix} \arrow[d, "\text{rescaling basis}"] \\ & \text{remove denominators and } \underline{\text{repeated}} \text{ irred. factors} \end{tikzcd}$$

$A \longmapsto P^T A P$

So assume $W_F = \mathrm{diag}(d_1, \dots, d_{2g})$ $\qquad d_i \in \mathbb{Q}[t^{\pm}]$

symmetric

product of dist. irred. factors.

<u>Lem</u> K knot

δ irred. symmetric poly. $\quad \alpha_i, \dots \alpha_j$ roots of δ on S^1

$$J_K(\alpha_i) = \sigma_K(\alpha_j) \qquad \forall i, j.$$

$$= J_K(\alpha_i) \qquad (\text{mod } 2)$$

<u>pf</u> $\quad W_F = \mathrm{diag}(f_1\delta, \dots, f_m\delta, g_1, \dots, g_n) \qquad m + n = 2g$ + (even)

$f.\ g.$ coprime to δ

$J_K(\alpha_i) = m \pmod 2$

$\sigma_K(\alpha_j) = n \pmod 2$

α_i <u>not</u> a root of $f.$, $g.$

$$\Rightarrow J_K(\alpha_i) + \sigma_K(\alpha_j) = m + n = 0 \pmod 2.$$

//.

$\boxed{K_+ \xrightarrow{+\text{ to }-} K_-}$

$\boxed{\begin{array}{l} \delta \text{ irred } \mathbb{Q}[t] \\ \alpha_1 \dots \alpha_j \text{ roots of } \delta \text{ on } S^1 \end{array}}$

Seifert alg. $\longrightarrow$ $\exists$ Seifert forms V_{F_+}, V_{F_-} identical except $V_{F_-}^{2g,2g} - V_{F_+}^{2g,2g} = 1.$

$$\xrightarrow{\text{lin. alg.}} \quad W_{F_\pm} = \operatorname{diag}(f_1\delta \dots f_m\delta, g_1 \dots g_n) \oplus \begin{pmatrix} a & \bar{b} \\ b & d + \underbrace{(1-t)(1-t^{-1})}_{\text{only for } W_{F_-}} \end{pmatrix}$$

f, g coprime to δ.

<u>Lemma</u>. Either (1) $\forall \alpha_i$: $J_{K_-}(\alpha_i) - J_{K_+}(\alpha_i) = 0$ $\quad \sigma_{K_-}(\alpha_i) - \sigma_{K_+}(\alpha_i) \in \{0,2\}$

or (2) $\forall \alpha_i$: $J_{K_-}(\alpha_i) - J_{K_+}(\alpha_i) = 1$ $\quad \sigma_{K_-}(\alpha_i) - \sigma_{K_+}(\alpha_i) = 1$

or (3) $\forall \alpha_i$: $J_{K_-}(\alpha_i) - J_{K_+}(\alpha_i) = -1$ $\quad \sigma_{K_-}(\alpha_i) - \sigma_{K_+}(\alpha_i) = 1$

<u>proof</u> — analysis of how $(1-\alpha_i)(1-\alpha_i^{-1})$ affects the signature .... //.

<u>Cor</u>. Either (1) $J_{K_-} - J_{K_+} = 0$ $\quad \underline{\sigma}_{K_-} - \underline{\sigma}_{K_+} \in \{0,2\}$

$\overline{\sigma}_{K_-} - \overline{\sigma}_{K_+} \in \{0,2\}.$

or (2) $J_{K_-} - J_{K_+} \in \{\pm 1\}$ $\quad \underline{\sigma}_{K_-} - \underline{\sigma}_{K_+} = 1$

$\overline{\sigma}_{K_-} - \overline{\sigma}_{K_+} = 1.$

//.

Step 2.

Thm $u_-(K) \geqslant \frac{1}{2}(J + \overline{\mathfrak{G}})$

$u_+(K) \geqslant \frac{1}{2}(J - \underline{\mathfrak{G}})$

proof $\Lambda = \{(x,y,z) \in \mathbb{Z}^3 \mid x \equiv y \equiv z \mod 2\}$.

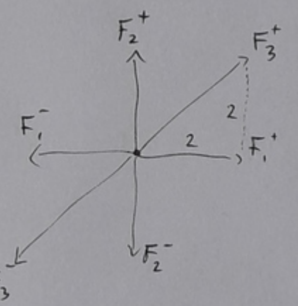
Automorphisms:

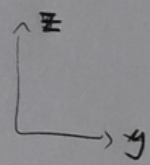
G_1^- : $(-1,-1,-1)$

G_2^- : $(+1,-1,-1)$

G_1^+ : $(-1,+1,+1)$

G_2^+ : $(+1,+1,+1)$





$K \longrightarrow U$

induces

$(J, \underline{\mathfrak{G}}, \overline{\mathfrak{G}}) \longrightarrow (0,0,0)$ via these automorphisms.

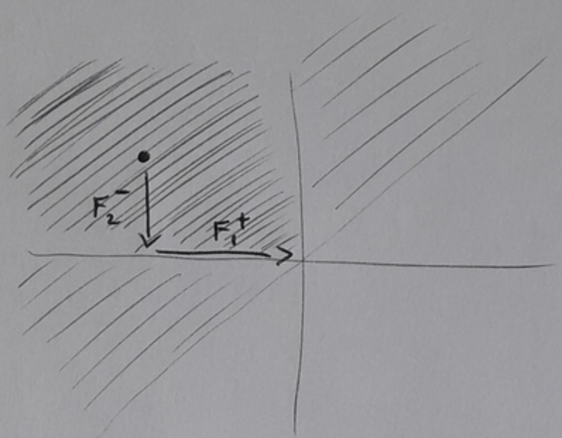
Suppose we have a <u>minimal</u> such sequence. (w.r.t. # of + functions)

wlog
- J instances of G_1^-, G_1^+
- then only F-types.

suppose a instances of G_1^-

$J-a$ —— " —— G_1^+ $\quad (0 \leq a \leq J_-)$

$$\longrightarrow \quad (0,\ \underline{Б} + J - 2a\ ,\ \overline{Б} + J - 2a)$$



$$u_-(K) \geqslant a + \tfrac{1}{2}(\overline{Б} + J - 2a) = \tfrac{1}{2}(J + \overline{Б})$$

$$u_+(K) \geqslant (J-a) + \tfrac{1}{2}(-(\underline{Б} + J - 2a)) = \tfrac{1}{2}(J - \underline{Б}).$$ //.

BONUS — some more detailed proofs, which I didn't cover in the talk

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$\boxed{\text{page 9}}$ proof of Lemma (more details)

$$W_F \approx \mathrm{diag}(f_1\delta, \ldots, f_m\delta, g_1, \ldots, g_n) \qquad m+n = 2g$$

$f_r, g_r \in \mathbb{Q}[t^{\pm}]$ coprime to δ.

- diagonalising
- rescaling basis to remove (i) denominators
  (ii) pairs of repeated roots
- permuting the basis

Obs

- $\left.\begin{array}{l} f_r \text{ coprime to } \delta \\ \alpha_i \text{ root of } \delta \end{array}\right] \longrightarrow \alpha_i$ not a root of f_r

  $\mathbb{Q}[t^{\pm}]$ is a P.I.D., so can write
  $1 = af_r + b\delta$

- (same for g_r)

- δ irreducible $\xrightarrow{\mathrm{char}(\mathbb{Q})=0}$ it has distinct roots in its splitting field, hence also in $\mathbb{C}$

  $\longrightarrow \alpha_i$ is a simple root of δ

  $\longrightarrow$ considering δ as a function $S^1 \longrightarrow \mathbb{R}$, the sign of $\delta(\alpha)$ changes as α goes from $\alpha_i - \varepsilon$ to $\alpha_i + \varepsilon$.

Using these observations, we see that:

$$\left(\text{abbreviating} \quad \begin{array}{l} \lim_{s\to t^+} f(s) = f(t^+) \\ \lim_{s\to t^-} f(s) = f(t^-) \end{array} \right)$$

$$\sigma_\kappa(\alpha_j) = \frac{1}{2}\left(\sigma_\kappa(\alpha_j^+) + \sigma_\kappa(\alpha_j^-)\right)$$

$$= \frac{1}{2}\left(\left[\sum_{i=1}^{m} \varepsilon_i + \sum_{i=m+1}^{m+n} \varepsilon_i\right] + \left[\sum_{i=1}^{m} (-\varepsilon_i) + \sum_{i=m+1}^{m+n} \varepsilon_i\right]\right)$$

cancel.

$\varepsilon_i \in \{\pm 1\}$.

$$= \sum_{i=m+1}^{m+n} \varepsilon_i$$

$$= n \pmod 2$$

$$J_\kappa(\alpha_i) = \frac{1}{2}\left(\sigma_\kappa(\alpha_i^+) - \sigma_\kappa(\alpha_i^-)\right)$$

$$= \frac{1}{2}\left(\left[\sum_{k=1}^{m} \bar{\varepsilon}_k + \sum_{k=m+1}^{m+n} \bar{\varepsilon}_k\right] - \left[\sum_{k=1}^{m} (-\bar{\varepsilon}_k) + \sum_{k=m+1}^{m+n} \bar{\varepsilon}_k\right]\right)$$

cancel.

$\bar{\varepsilon}_k \in \{\pm 1\}$.

$$= \sum_{k=1}^{m} \bar{\varepsilon}_k$$

$$= m \pmod 2$$

So $\sigma_\kappa(\alpha_j) + J_\kappa(\alpha_i) = m + n = 2g = 0 \pmod 2$ $\square$.

Rephrasing: if $\alpha_1 \ldots\ldots \alpha_j \in S'$ are all roots of the <u>same</u> irred. rational polynomial

then $\left\{\begin{matrix}\sigma_\kappa(\alpha_1), \ldots\ldots, \sigma_\kappa(\alpha_j) \\ J_\kappa(\alpha_1), \ldots\ldots, J_\kappa(\alpha_j)\end{matrix}\right\}$ <u>all have the same parity</u>.

page 10

proof of lemma

If $\underline{a=0}$ $\longrightarrow$ the last 2×2 block is ~ 0
so its signature and jump are 0

$$\Rightarrow \quad \sigma_{K_-} = \sigma_{K_+} \quad \text{and} \quad J_{K_-} = J_{K_+}$$

$\Rightarrow$ case (1) holds.

If $\underline{a\neq 0}$ $\longrightarrow$ we can fully diagonalise $W_{F_\pm}$, with the only difference between W_{F_-} and W_{F_+} being that

- the last entry of W_{F_+} is d
- the last entry of W_{F_-} is $d+(1-t)(1-t^{-1})$

$\longrightarrow$ just need to compare how $\left\{\begin{matrix} d \\ d+(1-t)(1-t^{-1}) \end{matrix}\right\}$ contribute to the signature and jump.

At a particular root α_i:

- The possible contributions of $\left\{\begin{matrix} d \\ d+(1-t)(1-t^{-1}) \end{matrix}\right\}$ to (|jump|, signature) are $\left\{\begin{matrix} (0,+1) \\ (1,0) \\ (0,-1) \end{matrix}\right\}$.

(This is true for any single entry in a diagonal matrix.)

- Moreover, $(1-\alpha)(1-\alpha^{-1}) \geq 0$ for $\alpha\in S^1$, so the signature cannot decrease from W_{F_+} to W_{F_-}. Thus the only possible non-trivial changes are:

$$\begin{matrix} & (0,+1) & \\ ① & (1,0) & ③ \\ & (0,-1) & ② \end{matrix}$$

(denote by ⓪ any trivial change)

We need to show that either

$$\left.\begin{array}{lll} & (1) \; \forall \alpha_i & \text{changes of type ⓪ or ① occur} \\ \text{or} & (2) \; \forall \alpha_i & \text{changes of type ② occur} \\ \text{or} & (3) \; \forall \alpha_i & \text{changes of type ③ occur.} \end{array}\right\} \circledast$$

To see this, write $d = f\delta^{\varepsilon_+}$

$$d + (1-t)(1-t^{-1}) = g\delta^{\varepsilon_-}$$

$\varepsilon_+, \varepsilon_- \in \{0,1\}$
f, g coprime to δ.

<u>Obs</u>: The contribution of this entry of $W_{F_\pm}$ to $|\text{jump}|$ <u>at any root of δ</u> is exactly $\varepsilon_\pm$.

↳ The change in $|\text{jump}|$ is <u>the same for all α_i</u>.

$\Rightarrow$ ⊛ holds. $\square$.

<u>Note</u>: In the paper (at least on the arXiv) he states a slightly weaker version of the Lemma on page 10, combining cases (2) and (3). But his proof shows that the stronger version, as written on page 10, holds.

$\boxed{\text{page 10}}$ <u>proof of Corollary</u>

This is immediate from the Lemma above, <u>plus</u> the fact (from page 9) that all $\sigma_K(\alpha_1), \ldots\ldots, \sigma_K(\alpha_j)$ have the <u>same</u> parity.

$\boxed{\text{page 11}}$ <u>proof of Theorem</u> (more detailed version)

Fix $\delta, \alpha_1, \ldots, \alpha_j$.

$$\{\text{knots}\} \xrightarrow{(J, \underline{\mho}, \overline{\mho})} \Lambda = \{(x,y,z) \in \mathbb{Z}^3 \mid x = y = z \mod 2\}$$

$$+ \text{ to } - \text{ crossing change} \longmapsto \text{operations of type } G_1^+, G_2^+, F_1^+, F_2^+, F_3^+$$

$$- \text{ to } + \text{ crossing change} \longmapsto \text{operations of type } G_1^-, G_2^-, F_1^-, F_2^-, F_3^-$$

pictured on page 11

(note: $F_i^{\pm}$ do not change the x-coordinate).

$\boxed{\text{by the Corollary on page 10}}$

$$\text{unknot} \longmapsto (0,0,0).$$

Need to find a lower bound on the minimal <u>length</u> of a sequence of these operations that takes $(J, \underline{\mho}, \overline{\mho})$ to $(0,0,0)$.

Where <u>length</u> $= \begin{cases} \text{either } \# \text{ of } G_i^+ \text{ and } F_i^+ \text{ in the sequence} \\ \text{or } \# \text{ of } G_i^- \text{ and } F_i^- \text{ in the sequence.} \end{cases}$

This will be a lower bound on the # of $\left\{\begin{matrix} + \text{ to } - \\ - \text{ to } + \end{matrix}\right\}$ crossing changes

needed to turn K into $\underbrace{\text{a knot with } (J, \underline{\sigma}, \overline{\sigma}) = (0,0,0)}$

$$\Uparrow$$

$$\underbrace{\text{a knot with trivial signature function}}$$

$$\Uparrow$$

$$\underbrace{\text{a slice knot}}$$

$$\Uparrow$$

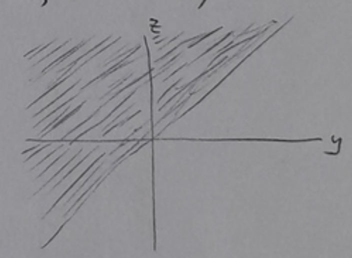
$$\underbrace{\text{the unknot}}$$

Suppose we have such a minimal length sequence.

- Since the 10 operations all commute, and $J \geq 0$, we may assume that the sequence is
  - a sequence of J instances of G_1^-, G_1^+ $\longrightarrow (0, *, *)$ [yz-plane]
  - <u>no</u> G_2^-, G_2^+ [may be paired with $G_1^\pm$ and replaced by $F_i^\pm$ of the same length]
  - a sequence of $F_i^\pm$ $\longrightarrow (0,0,0)$.

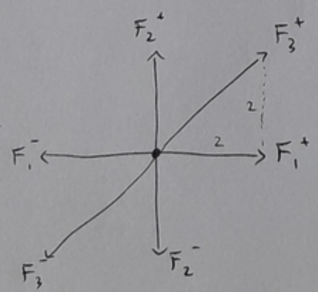
- Let's say we use n instances of G_1^-
  $J-n$ ——— G_1^+

$$\longrightarrow (0, \underline{\sigma} + J - 2n, \overline{\sigma} + J - 2n) \in$$



(we'll minimise over possible $0 \leq n \leq J$ later)

and then use

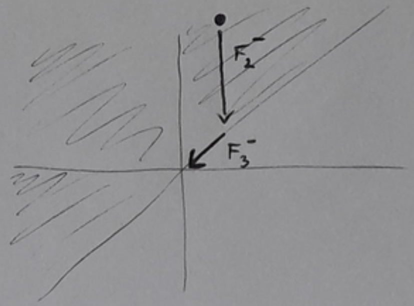


to move to $(0,0,0)$.

If $\boxed{\text{length} = \#\text{ of } G_i^+ \text{ and } F_i^+}$

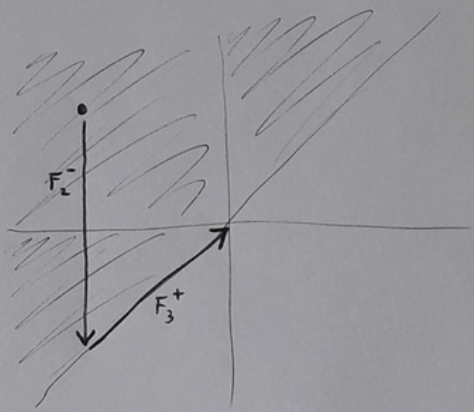
- Minimal length sequence of $F_i^\pm$ to reach $(0,0,0)$ is :

(if $\underline{\mathfrak{S}} + J - 2n \geqslant 0$)



length $= 0$

(if $\underline{\mathfrak{S}} + J - 2n \leqslant 0$)



length $= -\frac{1}{2}\left(\underline{\mathfrak{S}} + J - 2n\right)$

[Note: this is not the only minimal length sequence.]

- Total length $= \begin{cases} J - n + 0 \geqslant J - \frac{1}{2}(\underline{\mathfrak{S}} + J) = \frac{1}{2}(J - \underline{\mathfrak{S}}) & \text{if } \underline{\mathfrak{S}} + J - 2n \geqslant 0 \\ J - n - \frac{1}{2}(\underline{\mathfrak{S}} + J - 2n) = \frac{1}{2}(J - \underline{\mathfrak{S}}) & \text{if } \underline{\mathfrak{S}} + J - 2n \leqslant 0 \end{cases}$

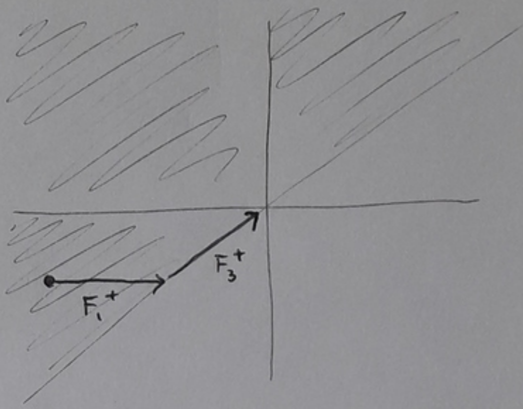
So the min. length is $\geqslant \frac{1}{2}(J - \underline{\mathfrak{S}})$ [indep. of n]

So $\underline{u_+(K) \geqslant \frac{1}{2}(J - \underline{\mathfrak{S}})}$.

If $\boxed{\text{length} = \# \text{ of } G_i^- \text{ and } F_i^-}$

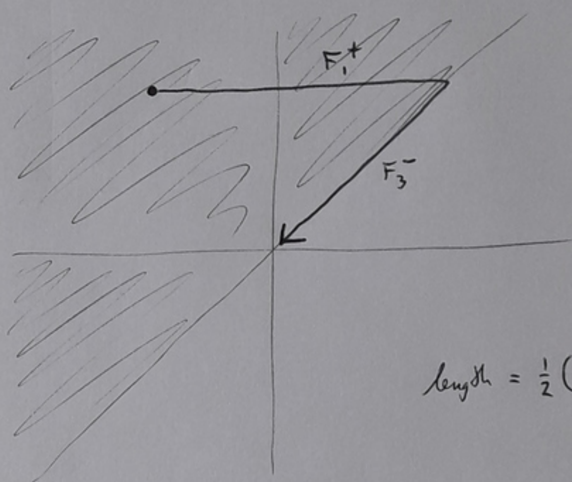
- Minimal length sequence of $F_i^{\pm}$ to reach $(0,0,0)$ is:

$\left(\text{if } \overline{\mathfrak{S}} + J - 2n \leq 0\right)$



length = 0

$\left(\text{if } \overline{\mathfrak{S}} + J - 2n \geq 0\right)$



length $= \frac{1}{2}\left(\overline{\mathfrak{S}} + J - 2n\right)$

- Total length $= \begin{cases} n + 0 \quad \geq \quad \frac{1}{2}\left(\overline{\mathfrak{S}} + J\right) & \text{if } \overline{\mathfrak{S}} + J - 2n \leq 0 \\ n + \frac{1}{2}\left(\overline{\mathfrak{S}} + J - 2n\right) = \frac{1}{2}\left(\overline{\mathfrak{S}} + J\right) & \text{if } \overline{\mathfrak{S}} + J - 2n \geq 0 \end{cases}$

So the min. length is $\geq \frac{1}{2}\left(\overline{\mathfrak{S}} + J\right)$ [indep. of n]

So $u_-(K) \geq \frac{1}{2}\left(\overline{\mathfrak{S}} + J\right)$.

$\square$.