Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Kupers and Randal-Williams

Martin Palmer-Anghel — GeMAT seminar, IMAR — 7/13/28 May and 4/11 June 2021

Abstract.

This will be an expository series of 5 talks on recent advances in understanding moduli spaces of discs in dimensions $\geqslant 5$, due to T. Watanabe, M. Weiss, A. Kupers and O. Randal-Williams.

The objects of study are the spaces $\mathcal{M}(\mathbb{D}^d)$ for a fixed dimension d. This is the moduli space of smooth manifolds M that are homeomorphic to the standard d-disc $\mathbb{D}^d$ by a homeomorphism that is a diffeomorphism near the boundary. In dimensions $\leqslant 3$ this space is contractible, by theorems of Radó, Moise, Smale and Hatcher. In contrast, in high dimensions, it is very complicated: for example, in dimensions $d \geqslant 7$ its loop space $\Omega\mathcal{M}(\mathbb{D}^d)$ is not homotopy equivalent to *any* finite CW-complex, by a theorem of Antonelli-Burghelea-Kahn.[1]

In these talks, we will focus on the higher (rational) homotopy groups of $\mathcal{M}(\mathbb{D}^d)$, for dimensions $d \geqslant 5$. The classical approach is quite indirect, using homotopy theory, surgery theory and pseudoisotopy theory, and works only in relatively low degrees (compared to the dimension) – the result is a calculation (Farrell-Hsiang) of the rational homotopy groups $\pi_i(\mathcal{M}(\mathbb{D}^d)) \otimes \mathbb{Q}$ in approximately the range $0 \leqslant i \leqslant \frac{d}{3}$.

I will talk about two more recent approaches – one due to T. Watanabe, and another discovered by M. Weiss and developed further by A. Kupers and O. Randal-Williams – which are both more direct and also work in much higher degrees. In fact, they work, in a sense, in complementary ranges of degrees, so there is hope that in combination they could potentially compute *all* rational homotopy groups of the moduli spaces $\mathcal{M}(\mathbb{D}^d)$. The first approach uses the Kontsevich graph complex, whereas the second uses manifold calculus and homological stability. The aim will be to give an overview of the ideas underlying these two different approaches.

[1] If one looks at spheres instead of discs, then the path-components $\pi_0(\mathcal{M}(\mathbb{S}^d))$ are in bijection with the group Θ_d of homotopy d-spheres, which is non-trivial for infinitely many d, and closely related to the stable homotopy groups of spheres, by a theorem of Kervaire and Milnor.