Representations of the Torelli group via the Heisenberg group

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IMAR

20 May 2021
Workshop for Young Researchers in Mathematics – 10th ed.

Joint work with Christian Blanchet and Awais Shaukat

Partially supported by project n° PN-III-P4-ID-PCE-2020-2798 (CNCS – UEFISCDI)

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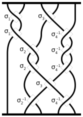
- {  }

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- [Moishezon]: alg. curve in $\mathbb{CP}^2 \longmapsto$ *braid monodromy* $F_N \to B_d$
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Homotopy theory:

- [Berrick-Cohen-Wong-Wu, 2006]:

$$\pi_*(S^2) \;\cong\; \frac{\{\text{Brunnian braids in } S^2 \times [0,1]\}}{\{\text{Brunnian braids in } D^2 \times [0,1]\}}$$

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- 4-dimensional symplectic topology [Donaldson]

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- Q: *Are the braid groups linear?*
  — Does B_n embed into some $GL_N(\mathbb{F})$?

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$$\mathrm{Lawrence}_k\colon B_n \longrightarrow GL_N(\mathbb{Z}[Q]) = \mathrm{Aut}_{\mathbb{Z}[Q]}\left(H_*^{bm}(C_k(D_n);\mathbb{Z}[Q])\right)$$

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Theorem [Bigelow'00,Krammer'00]

Lawrence_2 is faithful (injective). Hence B_n embeds into $GL_N(\mathbb{R})$.

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Main result [Blanchet-P.-Shaukat'21]

A new representation of (a central extension of) $\mathrm{Tor}(\Sigma) \subset \mathrm{Map}(\Sigma)$.

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- What is this?

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- Lower central series: $\pi_1(\Sigma) = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \cdots$
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Corollary [Moriyama'07]

$\bigoplus_{k=1}^{\infty} H_k^{bm}(F_k(\Sigma');\mathbb{Z})$ is a faithful (∞-dim.!) $\mathrm{Map}(\Sigma)$-representation.

Rep. of MCGs – abelian twisted coefficients

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Fact ($k \geqslant 2$)

$$B_k(S)^{ab} \cong \pi_1(S)^{ab} \oplus \begin{Bmatrix} \mathbb{Z} & S \text{ planar} \\ \mathbb{Z}/(2k-2) & S = S^2 \\ \mathbb{Z}/2 & \text{otherwise.} \end{Bmatrix}$$

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$$B_k(S)^{ab} \cong \pi_1(S)^{ab} \oplus \begin{Bmatrix} \mathbb{Z} & S \text{ planar} \\ \mathbb{Z}/(2k-2) & S = S^2 \\ \mathbb{Z}/2 & \text{otherwise.} \end{Bmatrix}$$

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Rep. of MCGs – abelian twisted coefficients

- Idea: Enrich the representation by taking homology with *twisted coefficients* $\mathbb{Z}[Q]$, where $\pi_1(C_k(\Sigma')) = B_k(\Sigma) \twoheadrightarrow Q$.
- $Q = \mathfrak{S}_k$ corresponds to the Moriyama representations:
$$H_k^{bm}(F_k(\Sigma'); \mathbb{Z}) = H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathfrak{S}_k]).$$
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- If S is non-planar, we can only "count" the *self-winding number* of braids on S **mod 2**. (or mod $2k-2$ if $S = S^2$)
- In $\mathbb{Z}[B_k(S)^{ab}]$, the corresponding "variable" t will have order two: $t^2 = 1$.
$\longmapsto$ We get a much "weaker" representation...

Rep. of MCGs – non-abelian twisted coefficients

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Theorem [Bellingeri'04]

$$B_k(\Sigma_{g,1}) \cong \left\langle \sigma_1, \ldots, \sigma_{k-1}, \begin{matrix} a_1, \ldots, a_g \\ b_1, \ldots, b_g \end{matrix} \;\middle|\; \cdots \text{ some relations } \cdots \right\rangle$$

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This is the *genus-g discrete Heisenberg group*. Note that:

$$\mathcal{H}_1 \cong \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3(\mathbb{Z})$$

Rep. of MCGs – non-abelian twisted coefficients

Lemma

The action $\mathrm{Map}(\Sigma) \circlearrowleft B_k(\Sigma)$ descends to a well-defined action on the quotient $\mathcal{H}_g$.

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Rep. of MCGs – non-abelian twisted coefficients

Proposition [Blanchet-P.-Shaukat'21]

(a) $\ker(\mathrm{Map}(\Sigma) \circlearrowleft \mathcal{H}_g) = \mathrm{Chill}(\Sigma)$

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Rep. of MCGs – non-abelian twisted coefficients

Theorem [Blanchet-P.-Shaukat'21]

We obtain well-defined representations, defined over $\mathbb{Z}[\mathcal{H}_g]$:

(a) $\mathrm{Chill}(\Sigma) \quad \circlearrowleft \quad H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$

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"Twisted" means:

- the action is $\mathbb{Z}$-linear
- there is also an action on the ground ring $\mathbb{Z}[\mathcal{H}_g]$
- these are compatible: $\varphi(\lambda.v) = \varphi(\lambda).\varphi(v)$.

Rep. of MCGs – non-abelian twisted coefficients

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Untwisting Lemma: A twisted representation of Γ over $\mathbb{Z}[G]$, where the action $\Gamma \circlearrowleft G$ is by conjugations, can be *untwisted* by passing to a central extension of Γ by $\mathcal{Z}(G) = \mathrm{Centre}(G)$.

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Q′: Is it smaller than $\mathfrak{J}(k) = \ker(\mathrm{Moriyama}_k)$?

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Corollary [Blanchet-P.-Shaukat'21]

The kernel of the $\widetilde{\mathrm{Tor}}(\Sigma)$-representation $H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$ is **strictly smaller** than $\mathfrak{J}(2)$.

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$$\begin{bmatrix}
\begin{matrix}\sigma^{-8}b^2+\sigma^{-4}a^{-2}-\sigma a^{-2}b^2+(\sigma^{-1}-\sigma^{-2})a^{-2}b+\\ (\sigma^{-3}-\sigma^{-4})a^{-1}b^2+(\sigma^{-4}-\sigma^{-5})a^{-1}b\end{matrix} &
\begin{matrix}(\sigma^2+1-2\sigma^{-1}+\sigma^{-2}+\sigma^{-4})a^{-2}b^2-\sigma a^{-2}b^4+\\ (-\sigma^2+\sigma+\sigma^{-1}-\sigma^{-2})a^{-2}b^3-\sigma^{-3}a^{-2}+\\ (-1+\sigma^{-1}+\sigma^{-3}-\sigma^{-4})a^{-2}b\end{matrix} &
\begin{matrix}(-1+2\sigma^{-1}-\sigma^{-2}-\sigma^{-4}+\sigma^{-5})a^{-2}b+(\sigma-1)a^{-2}b^3+\\ (\sigma^2-\sigma-\sigma^{-1}+2\sigma^{-2}-\sigma^{-3})a^{-2}b^2+(-\sigma^{-3}+\sigma^{-4})a^{-1}b+\\ (\sigma^{-4}-\sigma^{-5})a^{-1}b^3+(-\sigma^{-2}+\sigma^{-3}+\sigma^{-5}-\sigma^{-6})a^{-1}b^2+(-\sigma^{-3}+\sigma^{-4})a^{-2}\end{matrix} \\
\begin{matrix}-\sigma^{-1}-\sigma^{-3}+2\sigma^{-4}-\sigma^{-5}-\sigma^{-7}+\sigma^{-2}a^2+\\ (\sigma^{-1}-\sigma^{-2}-\sigma^{-4}+\sigma^{-5})a+\sigma^{-6}a^{-2}+\\ (\sigma^{-3}-\sigma^{-4}-\sigma^{-6}+\sigma^{-7})a^{-1}\end{matrix} &
\begin{matrix}1+\sigma^{-2}-\sigma^{-3}+\sigma^{-6}+\sigma^{-6}a^{-2}b^2-\sigma^{-1}b^2+\\ (\sigma^{-3}-\sigma^{-4})a^{-1}b^2+(-1+\sigma^{-1}+\sigma^{-3}-\sigma^{-4})b+\\ (\sigma^{-2}-2\sigma^{-3}+\sigma^{-4}+\sigma^{-6}-\sigma^{-7})a^{-1}b-\sigma^{-5}a^{-2}+\\ (-\sigma^{-2}+\sigma^{-3}+\sigma^{-5}-\sigma^{-6})a^{-1}+(\sigma^{-5}-\sigma^{-6})a^{-2}b\end{matrix} &
\begin{matrix}(-\sigma^{-6}+\sigma^{-7})a^{-2}b+(\sigma^{-1}-\sigma^{-2}-\sigma^{-4}+2\sigma^{-5}-\sigma^{-6})b+\\ (-\sigma^{-3}+2\sigma^{-4}-\sigma^{-5}-\sigma^{-7}+\sigma^{-8})a^{-1}b+1-\sigma^{-1}+\sigma^{-2}-\\ 3\sigma^{-3}+2\sigma^{-4}+\sigma^{-6}-\sigma^{-7}+(-\sigma^{-2}+2\sigma^{-3}-\sigma^{-4}+\sigma^{-5}-2\sigma^{-6}+\sigma^{-7})a^{-1}\\ +(\sigma^{-2}-\sigma^{-3})ab+(-1+\sigma^{-1}+\sigma^{-3}-\sigma^{-4})a+(-\sigma^{-5}+\sigma^{-6})a^{-2}\end{matrix} \\
\begin{matrix}-\sigma^{-6}ab+(-\sigma^{-3}+\sigma^{-4}-\sigma^{-7})b-\sigma^{-4}+\\ (\sigma^{-1}-\sigma^{-4}+\sigma^{-5})a^{-1}b+\sigma^{-2}a^{-2}b+\\ (-\sigma^{-3}+\sigma^{-6})a^{-1}+\sigma^{-5}a^{-2}\end{matrix} &
\begin{matrix}(-1-\sigma^{-2}+2\sigma^{-3}-\sigma^{-6})a^{-1}b+\sigma^{-1}a^{-1}b^3+\\ \sigma^{-2}a^{-2}b^3+(1-\sigma^{-1}-\sigma^{-3}+\sigma^{-4})a^{-1}b^2+\\ (\sigma^{-1}-\sigma^{-2}+\sigma^{-5})a^{-2}b^2+(-\sigma^{-1}+\sigma^{-4}-\sigma^{-5})a^{-2}b+\\ (\sigma^{-2}-\sigma^{-5})a^{-1}-\sigma^{-4}a^{-2}\end{matrix} &
\begin{matrix}\sigma^{-3}+(\sigma^{-2}-\sigma^{-3}-\sigma^{-5}+\sigma^{-6})a^{-1}+\\ (-\sigma^{-1}+\sigma^{-2}-\sigma^{-5}+\sigma^{-6})a^{-1}b^2+(-\sigma^{-2}+\sigma^{-3})a^{-2}b^2+\\ (-1+\sigma^{-1}+2\sigma^{-3}-3\sigma^{-4}+\sigma^{-7})a^{-1}b+\\ (-\sigma^{-1}+\sigma^{-2}-\sigma^{-5}+\sigma^{-6})a^{-2}b+(-\sigma^{-4}+\sigma^{-5})b^2+\\ (\sigma^{-2}-\sigma^{-3}-\sigma^{-5}+\sigma^{-6})b+(-\sigma^{-4}+\sigma^{-5})a^{-2}\end{matrix}
\end{bmatrix}$$

Thank you!