Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Kupers–Randal-Williams

GeMAT seminar
IMAR
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I. Introduction

Spaces of smooth structures

Fix
- a topological d-manifold M
- a smooth structure $\underbrace{\text{on a neighbourhood of}}_{\text{"near"}}$ ∂M (if $\partial M \neq \emptyset$)

The/an aim of differential topology is to understand

$$\mathrm{Sm}(M) = \{ \underbrace{\text{smooth structures}}_{\text{maximal } C^\infty \text{ atlas}} \text{ on } M \}$$

compatible with the given one near ∂M

as a topological space.

First Q: How do we topologise this?

$\mathrm{Sm}(M)$ has an action of $\mathrm{Homeo}(M)$ by pulling back smooth structures.

(!) slightly non-standard: "homeomorphism" will mean "homeo. that is a diffeo. near the boundary"

so by the orbit-stabiliser theorem we have an isomorphism of $\mathrm{Homeo}(M)$-sets

$$\mathrm{Sm}(M) \cong \coprod_{\substack{\{\text{orbits } [\mathcal{A}] \text{ of smooth} \\ \text{structures } \mathcal{A} \text{ on } M\}}} \mathrm{Homeo}(M) \big/ \mathrm{Diff}(M, \mathcal{A})$$

1:1

(exercise)

$$\left\{ \left\{ \begin{array}{l} \text{smooth manifolds } X \text{ that are} \\ \underbrace{\text{homeomorphic}}_{(\text{diffeo. near } \partial)} \text{ to } M \end{array} \right\} \Big/ \text{diffeomorphism} \right\} =: \mathrm{Smooth}(M)$$

$$\mathrm{Sm}(M) \cong \coprod_{[X] \in \mathrm{Smooth}(M)} \mathrm{Homeo}(X) \big/ \mathrm{Diff}(X)$$

compact-open topology

A more flexible model

$$\underline{\mathrm{Sm}}(M) := \left\{ \begin{array}{l} Y \subseteq \mathbb{R}^\infty \\ M \xrightarrow{f} Y \end{array} \;\middle|\; \begin{array}{l} Y \text{ smooth submanifold} \\ f \text{ homeomorphism} \end{array} \right\} \xleftarrow{\cong} \coprod_{[X] \in \mathrm{Smooth}(M)} \frac{\mathrm{Homeo}(M, X) \times \mathrm{Emb}(X, \mathbb{R}^\infty)}{\mathrm{Diff}(X)}$$

$$\begin{bmatrix} Y = e(X) \\ f = e \circ g \end{bmatrix} \longleftarrow\!\shortmid \left[M \xrightarrow{g} X \xhookrightarrow{e} \mathbb{R}^\infty \right]$$

$\boxed{\mathrm{Emb}(-) = \{\text{smooth embeddings}\} \text{ with Whitney topology}}$

<u>Fact:</u> $\underline{\mathrm{Sm}}(M) \twoheadrightarrow \mathrm{Sm}(M)$

$(Y, f) \longmapsto$ pull back smooth structure of Y along f

is a fibre bundle, and its fibres $(\mathrm{Emb}(X, \mathbb{R}^\infty))$ are contractible, so it is a homotopy equivalence.

The moduli space

$\underline{Sm}(M)$ has a free action of $Homeo(M)$:

The <u>moduli space of smooth manifolds homeomorphic to M</u> is

$$\mathcal{M}(M) := \underline{Sm}(M) / Homeo(M) = \left\{ Y \subseteq \mathbb{R}^\infty \;\middle|\; \begin{array}{l} Y \text{ smooth submanifold} \\ \text{s.t. } \exists \text{ homeo. } M \to Y \end{array} \right\}$$

$$\cong \coprod_{[X] \in Smooth(M)} Emb(X, \mathbb{R}^\infty) / Diff(X)$$

$$\cong \coprod_{[X] \in Smooth(M)} \underbrace{BDiff(X)}_{\substack{\text{classifying space} \\ \text{for smooth } X\text{-bundles}}}$$

We will be interested in the <u>mod-∂ version</u>:

First note that

$$Sm(M) \cong \coprod_{[X]} Homeo(X) / Diff(X) \cong \coprod_{[X]} Homeo_\partial(X) / Diff_\partial(X)$$

restricting to id. near the boundary

and similarly

$$\underline{Sm}(M) \cong \coprod_{[X]} \frac{Homeo(X) \times Emb(X, \mathbb{R}^\infty)}{Diff(X)} \cong \coprod_{[X]} \frac{Homeo_\partial(X) \times Emb(X, \mathbb{R}^\infty)}{Diff_\partial(X)}$$

$$\begin{array}{ccc} Diff_\partial(X) & \hookrightarrow & Homeo_\partial(X) \\ \downarrow & & \downarrow \\ Diff(X) & \hookrightarrow & Homeo(X) \end{array}$$

is a pushout of topological groups

Then we define:

$$\mathcal{M}_\partial(M) := \underline{\mathrm{Sm}}(M) / \mathrm{Homeo}_\partial(M) = \left\{ \begin{array}{l} Y \subseteq \mathbb{R}^\infty \text{ smooth submanifold} \\ \text{choice of a "germ" near } \partial M \text{ of} \\ \quad \text{a homeomorphism } M \to Y \end{array} \right\}$$

$$\cong \coprod_{[X] \in \mathrm{Smooth}(M)} \mathrm{Emb}(X, \mathbb{R}^\infty) / \mathrm{Diff}_\partial(X)$$

$$\cong \coprod_{[X] \in \mathrm{Smooth}(M)} \underbrace{B\mathrm{Diff}_\partial(X)}_{\substack{\text{classifying space for} \\ \text{smooth } X\text{-bundles whose} \\ \text{fibrewise boundary is the} \\ \text{trivial } \partial X\text{-bundle}}}$$

$\boxed{\text{Summary}}$

$$\begin{tikzcd}
\mathrm{Homeo}_\partial(M) \arrow[d] & \\
\underline{\mathrm{Sm}}(M) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Sm}(M) \cong \coprod_{[X]} \mathrm{Homeo}_\partial(X) / \mathrm{Diff}_\partial(X) \\
\mathcal{M}_\partial(M) \cong \coprod_{[X]} B\mathrm{Diff}_\partial(X) &
\end{tikzcd}$$

$\pi_0(\mathcal{M}_\partial(M)) = \mathrm{Smooth}(M) = \{\text{set of inequivalent smooth structures on } M\}$

$H^*(\mathcal{M}_\partial(M))$ = characteristic classes of smooth bundles with fibre M (with some smooth structure)

$\pi_*(\mathcal{M}_\partial(M))$ = isom. classes of smooth M-bundles over spheres

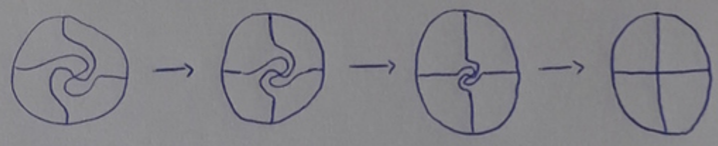
with some smooth str.

<u>Plan for today</u>:

- What is known for $M = D^d, S^d, \mathbb{R}^d$
- "Landscape" of calculations of $\mathrm{Diff}_\partial(D^d)$, $d \geq 5$

What is known for D^d?

First note that $\mathrm{Homeo}_\partial(D^d) \simeq *$ by <u>Alexander's trick</u>



$t=0$ $\quad$ $t=1$

Hence $\underline{\mathrm{Sm}(D^d) \simeq \mathcal{M}_\partial(D^d)}$

<u>Fact</u>: For $d \neq 4$, D^d has a unique smooth structure up to diffeomorphism.

$d=2$: Radó

$d=3$: Moise

$d \geqslant 6$: $\Leftarrow$ h-cobordism theorem (Smale)

$d=5$: Milnor

So $\mathcal{M}_\partial(D^d)$ is connected,

$\mathcal{M}_\partial(D^d) \cong B\mathrm{Diff}_\partial(D^d)$

For $d \leqslant 3$, $\mathrm{Diff}_\partial(D^d) \simeq *$ $\quad \left(\begin{array}{ll} d=2: & \text{Smale '59} \\ d=3: & \text{Hatcher '83} \end{array} \right)$

But $\mathrm{Diff}_\partial(D^4) \not\simeq *$ $\quad$ (Watanabe '18)

For $d \geqslant 5$, $\mathrm{Diff}_\partial(D^d)$ $\longrightarrow$ the main topic of these talks......

But first we briefly look at $M = S^d$

$M = \mathbb{R}^d$

What is known for S^d?

$$\mathrm{Sm}(S^d) = \coprod_{[\Sigma]\in\Theta_d} \mathrm{Homeo}(\Sigma)/\mathrm{Diff}(\Sigma) \qquad \mathcal{M}(S^d) = \coprod_{[\Sigma]\in\Theta_d} B\mathrm{Diff}(\Sigma)$$

So we need to understand the set Θ_d

& the topological groups $\mathrm{Diff}(\Sigma)$

$\mathrm{Homeo}(\Sigma) \cong \mathrm{Homeo}(S^d)$

- Θ_d is an <u>abelian group</u> under connected sum, if $d \neq 4$
  (Θ_4 is an abelian monoid)

  $\Theta_1 = \Theta_2 = \Theta_3 = *$ —— since all manifolds of dim ≤ 3 have a unique smooth structure up to diffeo.

  [Radó, Moise]
  $d=2$, $d=3$

  Θ_d is <u>finite</u> if $d \neq 4$ [Kervaire-Milnor]

  and fits into a short exact sequence

$$1 \to \boxed{\text{cyclic}} \to \Theta_d \to \boxed{\begin{array}{c}\text{subquotient}\\ \text{of } \pi_d(\mathbb{S})\end{array}} \to 1$$

- $\mathrm{Diff}(S^d) \simeq \mathrm{Diff}_\partial(D^d) \times \underbrace{O(d+1)}_{\text{"well-understood"}}$ [Antonelli-Burghelea-Kahn]

  So this boils down to understanding $\mathrm{Diff}_\partial(D^d)$ ......

- $\mathrm{Diff}(\Sigma) \simeq$?

- $\underline{\text{Homeo}(S^d)}$:

Fact: For all smooth manifolds M of dim ≤ 3,

$$\mathrm{Diff}(M) \hookrightarrow \mathrm{Homeo}(M)$$

is a homotopy equivalence.

$d=3$: [Cerf '59], assuming the $\underbrace{\text{Smale conjecture}}_{\text{[Hatcher '83]}}$

So for $d \leq 3$ we have:

$$\underset{\text{[Smale, Hatcher]}}{O(d+1) \xhookrightarrow{\simeq} \mathrm{Diff}(S^d)} \xhookrightarrow{\simeq} \mathrm{Homeo}(S^d)$$

But this certainly fails in higher dimensions, e.g.

$|\pi_0 \mathrm{Diff}(S^6)| = 56$

$|\pi_0 \mathrm{Homeo}(S^6)| = 2$

(*) **Fact**: $\boxed{\pi_0 \mathrm{Homeo}(S^d) \cong \mathbb{Z}/2}$ $\longleftrightarrow$

In contrast:

$$\pi_0 \mathrm{Diff}^+(S^d) \cong \Theta_{d+1}$$
$$[\varphi] \longmapsto D^{d+1} \cup_\varphi D^{d+1}$$

[Cerf / Smale]

Hence we have:

$$\pi_0 \mathcal{M}(S^d) \cong \Theta_d$$
$$\pi_1(\mathcal{M}(S^d), S^d) \underset{\text{index-2}}{\geqslant} \Theta_{d+1}$$

Let's prove this using the $\underline{\text{stable homeomorphism conjecture}}$

$$\mathrm{Homeo}^s(M) = \left\{ \varphi \in \mathrm{Homeo}(M) \;\middle|\; \begin{array}{l} \varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_k \\ \varphi_i|_{U_i} = \mathrm{id} \quad \text{for some} \\ \text{non-}\emptyset,\ \text{open } U_i \subseteq M \end{array} \right\}$$

Thm $\left(\overset{d=3}{\text{Fisher '60}},\ \overset{d\geq 5}{\text{Kirby '69}},\ \overset{d=4}{\text{Quinn '82}} \right)$

$$\mathrm{Homeo}^s(\mathbb{R}^d) = \mathrm{Homeo}^+(\mathbb{R}^d)$$

Proof of (*)

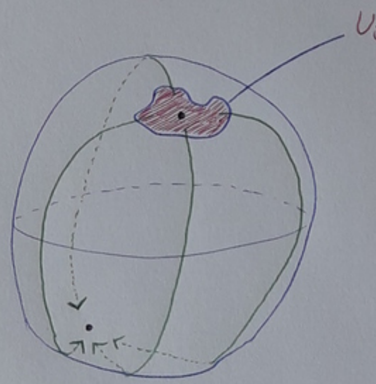
Let $\varphi \in \mathrm{Homeo}^+(S^d)$

- choose a stable homeo. φ_0 such that $\varphi_0^{-1} \circ \varphi$ fixes the north pole.
  so $\varphi_0^{-1} \circ \varphi$ restricts to $\in \mathrm{Homeo}^+(\mathbb{R}^d)$

- SHC $\Rightarrow$ $\underbrace{\varphi_0^{-1}}_{\text{stable}} \circ \varphi = \underbrace{\varphi_1 \circ \cdots \circ \varphi_k}_{\text{stable}} \in \mathrm{Homeo}^+(\mathbb{R}^d)$

- one-point-compactify $\longrightarrow$ $\varphi = \varphi_0 \circ \varphi_1 \circ \cdots \circ \varphi_k \in \mathrm{Homeo}^+(S^d)$

- $\underline{\varphi_i|_{U_i} = \mathrm{id}}$



Alexander's trick $\Rightarrow$ $\varphi_i \simeq \mathrm{id}$

hence $\varphi \simeq \mathrm{id}$. $\square$

What is known for $\mathbb{R}^d$?

- For $d \neq 4$, $\mathbb{R}^d$ has a unique smooth structure up to diffeomorphism
[Stallings, Munkres, Hirsch]

- By $\underbrace{\text{the opposite of the Alexander trick}}_{\text{"stretching"}}$, $O(d) \xhookrightarrow[\simeq]{} \mathrm{Diff}(\mathbb{R}^d)$.

So
$$\mathrm{Sm}(\mathbb{R}^d) \simeq \underbrace{\mathrm{Homeo}(\mathbb{R}^d) / O(d)}_{\boxed{\text{This also reduces to studying } \mathrm{Diff}_\partial(D^d)\text{, via...}}} \qquad \mathcal{M}(\mathbb{R}^d) \simeq \underbrace{BO(d)}_{\text{"well-understood"}}$$

Smoothing theory

- In general $(d \neq 4)$:

$$\mathrm{Sm}(M) \simeq \coprod_{[X]} \mathrm{Homeo}_\partial(X) / \mathrm{Diff}_\partial(X)$$

fibre over $p \in X$ is $\mathrm{Sm}(T_p X)$

$$\mathrm{Homeo}_\partial(X) / \mathrm{Diff}_\partial(X) \xrightarrow[\text{infinitesimally at each point}]{\text{remember the smooth str.}} \Gamma_\partial \begin{pmatrix} \mathrm{Sm}(TX) \\ \downarrow \\ X \end{pmatrix}$$

Fact: (Burghelea-Lashof '74, Hirsch-Mazur '74, Kirby-Siebenmann '77)

$$\boxed{\text{This is a homotopy equivalence onto a union of path-components.}}$$

("h-principle")

- Apply to $M = D^d$:

$$\mathrm{Homeo}_\partial(D^d)/\mathrm{Diff}_\partial(D^d) \xrightarrow{\ \sim\ } \mathrm{Map}_\partial\left(D^d, \mathrm{Sm}(\mathbb{R}^d)\right)$$

$$\mathrm{Homeo}_\partial(D^d)/\mathrm{Diff}_\partial(D^d) \cong B\mathrm{Diff}_\partial(D^d) \quad \text{(Alexander trick)}$$

$$= \mathrm{Map}_*\left(S^d, \mathrm{Homeo}(\mathbb{R}^d)/O(d)\right)$$

$$= \Omega^d\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d)\right)$$

(notation)

Coro: $\boxed{\mathrm{Diff}_\partial(D^d) \simeq \Omega^{d+1}\left(\mathrm{Homeo}(\mathbb{R}^d)/O(d)\right)}$

"Morlet's lemma".

Summary:

$d \leq 3$

	$\mathbb{R}^d$	D^d	S^d
$\mathrm{Smooth}(M)$	$*$	$*$	$*$
$\mathrm{Diff}_\partial(M)$	$O(d)$	$*$	$O(d+1)$
$\mathrm{Homeo}_\partial(M)$	same π_i as $O(d)$ for $i \geqslant d+1$	$*$	$O(d+1)$

$d \geqslant 5$

	$\mathbb{R}^d$	D^d	S^d
$\mathrm{Smooth}(M)$	$*$	$*$	Θ_d
$\mathrm{Diff}_\partial(M)$	$O(d)$	$\square$	$\square$
$\mathrm{Homeo}_\partial(M)$	$\square$	$*$	?

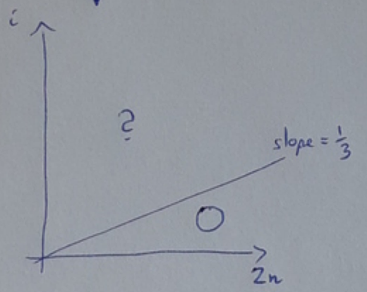
Morlet

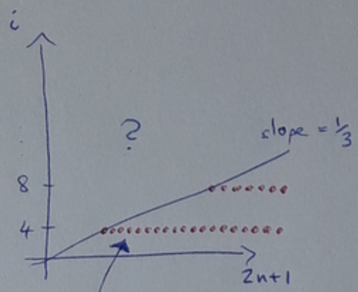
$\mathrm{Diff}_\partial(D^d) \times O(d+1) \simeq \mathrm{Diff}(S^d)$

"Landscape" of calculations of $\pi_i(\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q}$

Low degrees

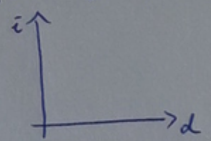
"Classical approach" (surgery theory + pseudoisotopy theory + algebraic K-theory) [Farrell-Hsiang '78, Igusa '84]





Graphs of

$\pi_i\, B\mathrm{Diff}_\partial(D^d) \otimes \mathbb{Q}$



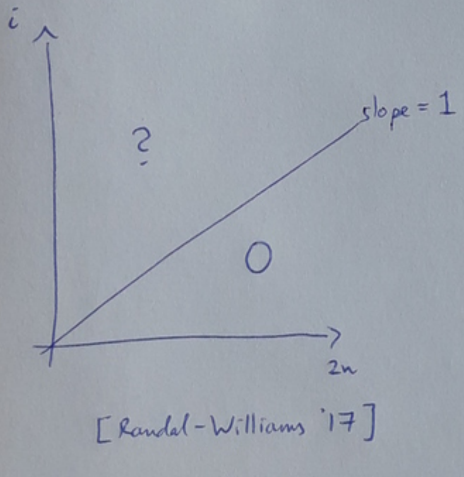
$d = 2n$
$d = 2n+1$

$\bullet = \mathbb{Q}$

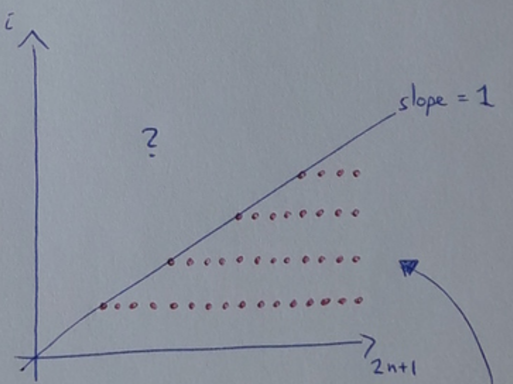
$$K_{i+1}(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & i \equiv 0 \pmod 4 \\ 0 & \text{o/w.} \end{cases}$$

[Borel '74]

More recently (high-dimensional topology)



[Randal-Williams '17]

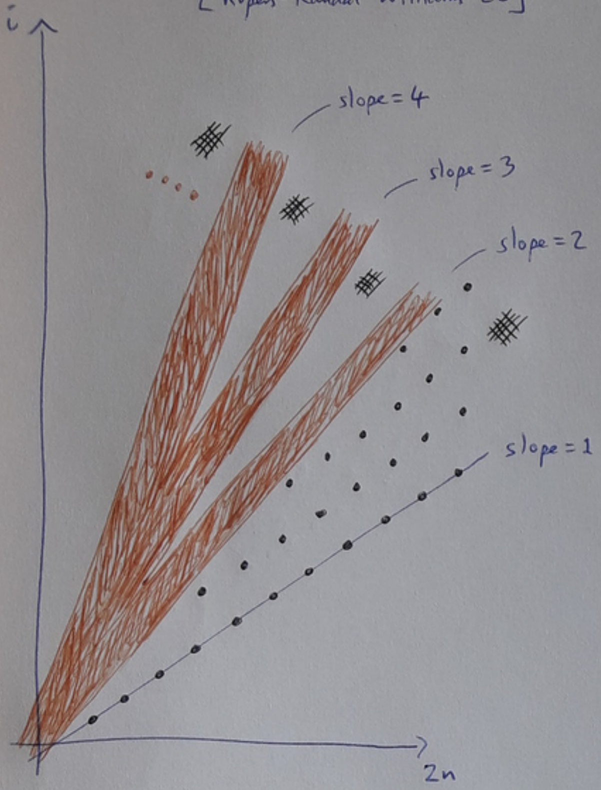


$K_{i+1}(\mathbb{Z}) \otimes \mathbb{Q}$

[Krannich '20]

High degrees

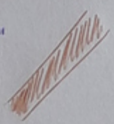
[Kupers-Randal-Williams '20]



 $= \begin{cases} \mathbb{Q} & i \equiv 2n-1 \pmod 4 \\ 0 & \text{o/w} \end{cases}$

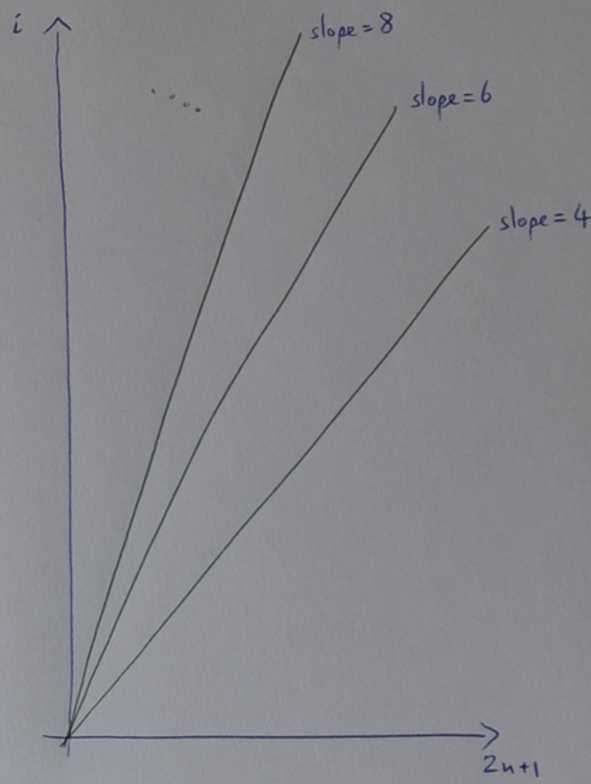
generated by $\underbrace{\text{"exotic Pontrjagin classes"}}$

discovered by [Weiss '15]

"bands"  = ?

- bands of <u>even</u> slope : exotic P. classes $\neq 0$ but there could be more
- bands of <u>odd</u> slope : ???

[Watanabe '09]



Each entry on a green line of slope 2k

$\downarrow$

$\mathcal{A}_k$ —— graph homology in degree k

Plan for next 4 talks:

~ 2 talks on [Watanabe '09] $\longrightarrow$ graph homology classes

~ 2 talks on [Weiss '15] + [Kupers-Randal-Williams '20] $\rightsquigarrow$ exotic Pontrjagin classes.