Moduli spaces of high-dimensional discs

after Watanabe, Weiss, Kupers-Randal-Williams

GeMAT seminar
IMAR
13 May 2021

II. Odd-dimensional discs, just above the stable range

0. Recollections

Aim: Study the homotopy type of $\mathrm{Diff}_\partial(D^d) = \left\{ D^d \xrightarrow{\varphi} D^d \;\middle|\; \begin{array}{l}\varphi \text{ diffeomorphism fixing} \\ \text{a neighbourhood of } \partial D^d\end{array} \right\}$

$d \leq 3$: $\mathrm{Diff}_\partial(D^d) \simeq *$ [Smale '59, Hatcher '83]

$d \geq 7$: $\mathrm{Diff}_\partial(D^d) \not\simeq$ any finite CW-complex [ABK] '72

$\|$

[Antonelli-Burghelea-Kahn]

In particular: $\boxed{\pi_*(\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q}}$ $\boxed{d = 2k+1}$

"Stable range": $\pi_i(\mathrm{Diff}_\partial(D^d)) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & i \equiv 3 \pmod 4 \\ 0 & \text{o/w} \end{cases}$

in the range $i \leq \frac{d-43}{6}$ [Farrell-Hsiang '78]

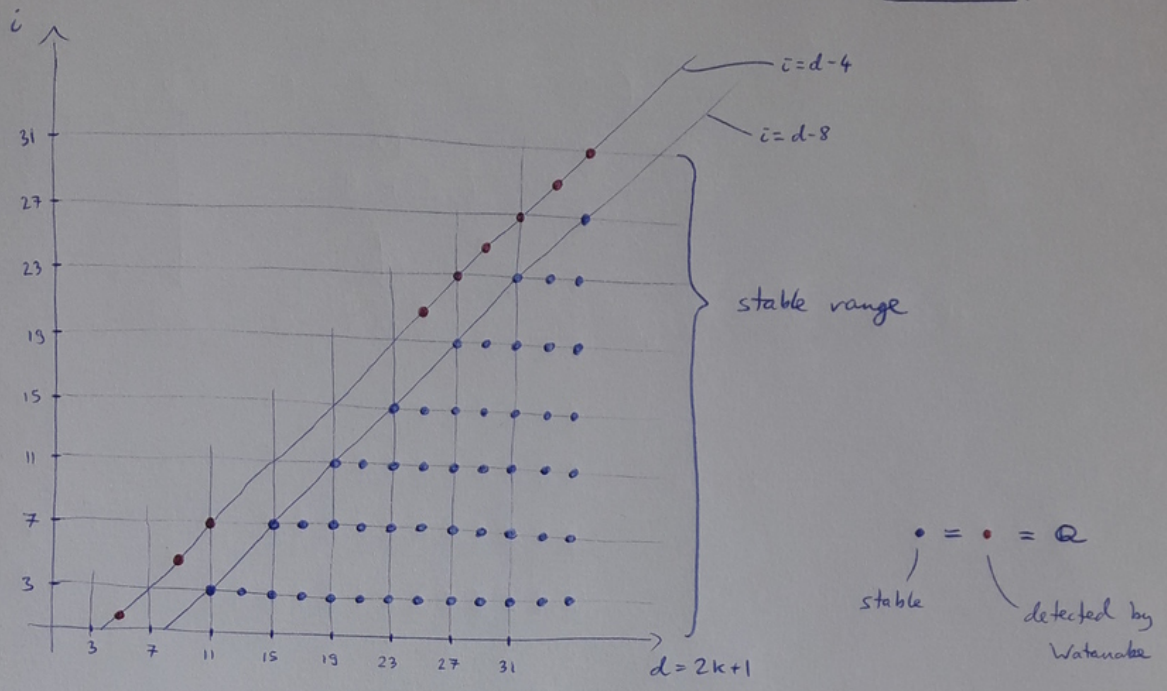
$i \leq \min\left(\frac{d-2}{3}, \frac{d-7}{2}\right)$ [Igusa '84]

$i \leq d-8$ [Krannich '20]

Just above the stable range:

[Watanabe '09] detects new elements of $\pi_i(\mathrm{Diff}_\partial(D^{2k+1})) \otimes \mathbb{Q}$

with $\underline{i = d-4}$.



(In the stable range, the elements • generate all of $\pi_*(-) \otimes \mathbb{Q}$.)

by building on work of [Antonelli-Burghelea-Kahn '72]

(*)
- [ABK] — construct elements of $\pi_*(\mathrm{Diff}_\partial(D^d))$ by "plumbing"
  — detect <u>non-triviality</u> via invariants with values in finite cyclic groups.
- [W] — detects <u>∞-order</u> by lifting these invariants (up to scalar) to $\mathbb{Z}$.

<u>Today</u>: Understand (*).

<u>Next time</u>: Look at [Watanabe '09 Part II]

→ elements on lines of higher slope.

Outline

1. The Growoll filtration
2. Plumbing ⇒ exotic spheres
3. Refinement of plumbing
4. The subgroup $bP_{n+1} \subseteq \Theta_n$
5. Detection results (torsion)
6. Infinite-type results
7. Watanabe's construction $\hat{Z}_2$
8. Detection results (non-torsion)

1. The Gromoll filtration

Def $\Theta_n = \{\text{smooth, closed, connected, oriented manifolds } \underline{\text{homeo}} \text{ to } S^n\} / \text{diffeomorphism}.$

Lemma $(n \neq 4)$ Θ_n is an abelian group w.r.t. $\#$

$\exists$ homomorphism
$$\begin{array}{ccc} \pi_0(\mathrm{Diff}^+(S^{n-1})) & \longrightarrow & \Theta_n \\ \cup\!\!\shortmid & & \cup\!\!\shortmid \\ \varphi & \longmapsto & D^n \cup_\varphi D^n \end{array}$$

Thm (Cerf, Smale) This is an <u>isomorphism</u> if $n \neq 4$.

Def (Novikov) $\Psi : \pi_i(\mathrm{Diff}_\partial(D^n)) \longrightarrow \pi_{i-j}(\mathrm{Diff}_\partial(D^{n+j}))$

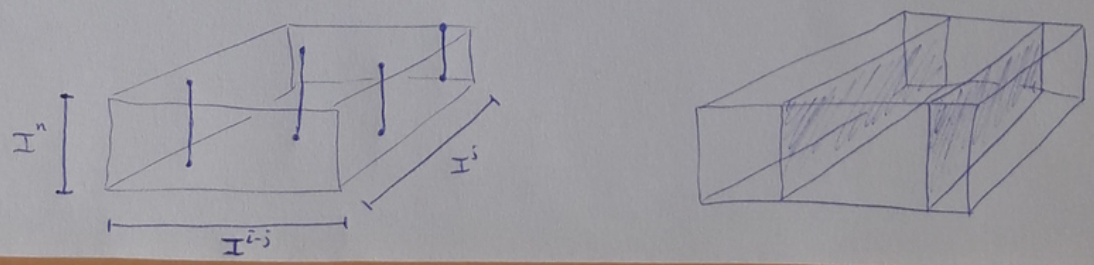
Idea of construction:

$$\begin{array}{ccc} \pi_i(\mathrm{Diff}_\partial(D^n)) & & \pi_{i-j}(\mathrm{Diff}_\partial(D^{n+j})) \\ \updownarrow & & \updownarrow \\ \varphi : I^i \longrightarrow \mathrm{Diff}(I^n) & \left[\begin{array}{l} \varphi(x) = id, \ x \in \partial I^i \\ \varphi(x)|_{\partial I^n} = id \end{array}\right] & \\ \updownarrow & & \\ \tilde{\varphi} \text{ diffeo of } I^i \times I^n & \longrightarrow & \tilde{\varphi} \text{ diffeo of } I^{i-j} \times I^{n+j} \end{array}$$

- id near the ∂
- preserves the partition $\{\{t\} \times I^n\}_{t \in I^i}$

- id near the ∂
- preserves the partition $\{\{s\} \times I^{n+j}\}_{s \in I^{i-j}}$

Picture for
$i = 2$
$n = 1$
$j = 1$



Def (Gromoll)

$$\Theta_n(i) := \text{image}\left(\pi_i \mathrm{Diff}_\partial(D^{n-1-i}) \xrightarrow{\Psi} \pi_0 \mathrm{Diff}_\partial(D^{n-1}) \cong \pi_0 \mathrm{Diff}^+(S^{n-1}) \xrightarrow{\cong} \Theta_n \right)$$

$$\boxed{\Theta_n = \Theta_n(0) \supseteq \Theta_n(1) \supseteq \cdots\cdots \supseteq \Theta_n(n-2) = 0}$$

Gromoll filtration.

In fact $\Theta_n(n-4) = 0$ [Hatcher]

$\Theta_n(0) = \Theta_n(1)$ [Cerf]

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2. Plumbing $\Rightarrow$ exotic spheres

Def (Milnor) $f: S^p \to SO(q+1)$

$g: S^q \to SO(p+1)$

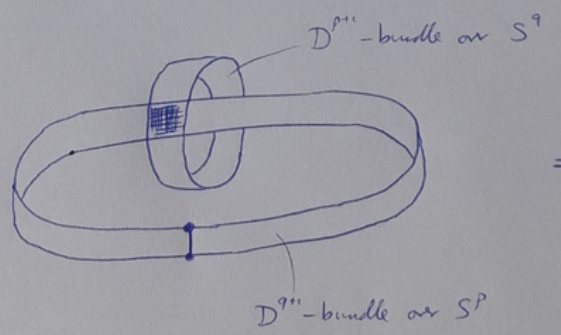
$$M(f,g) := \begin{array}{ccc} D^{p+1} \times D^{q+1} & & D^{p+1} \times D^{q+1} \\ \cup\!\!\mid & & \cup\!\!\mid \\ S^p \times D^{q+1} & & S^p \times D^{q+1} \\ & (x,y) \longleftrightarrow (x, f(x)(y)) & \end{array} \;=\; \begin{array}{ccc} D^{p+1} \times D^{q+1} & & D^{p+1} \times D^{q+1} \\ \cup\!\!\mid & & \cup\!\!\mid \\ D^{p+1} \times S^q & & D^{p+1} \times S^q \\ & (g(x)(y), x) \longleftrightarrow (y, x) & \end{array}$$

D^{q+1}-bundle over S^p ($\longleftrightarrow f$)

D^{p+1}-bundle over S^q ($\longleftrightarrow g$)

Picture:

(schematic)



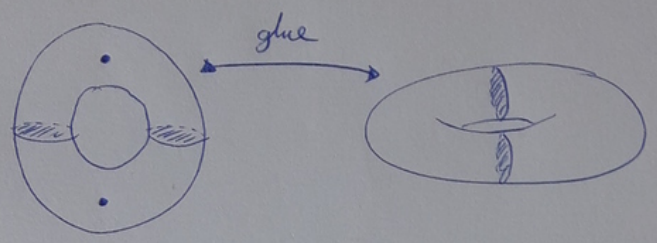
$= M(f,g)$

Its boundary looks like:

$$\begin{array}{ccc} D^{p+1} \times S^q & & D^{q+1} \times S^p \\ \cup\!\!\mid & & \cup\!\!\mid \\ S^p \times S^q & \longleftrightarrow & S^q \times S^p \end{array}$$

$$(x,y) \longmapsto \left(f(x)(y) \,,\, g(f(x)(y))^{-1}(x) \right)$$

<u>Another picture</u>: $(p=q=1)$



$= \partial M(f,g)$.

<u>Lemma</u>: $M(f,g)$ depends only on the <u>homotopy classes</u> of $f : S^p \to SO(q+1)$
$g : S^q \to SO(p+1)$

<u>Lemma</u> (Milnor) if $\text{image}(f) \subseteq SO(q)$ or $\text{image}(g) \subseteq SO(p)$ then $\partial M(f,g)$ is <u>homeo</u> to S^{p+q+1}.

<u>Idea</u>: Construct an explicit Morse function on $\partial M(f,g)$ with exactly 2 critical points.

- $D^{p+1} \times S^q \longrightarrow \mathbb{R}$
  $(x,y) \longmapsto \text{height}(y) \cdot (1 + |x|^2)^{-1/2}$] 2 critical pts.

- extend to $S^p \times D^{q+1}$ with <u>no</u> extra critical points. //

<u>Coro</u>. $\exists$ linear map

$$\pi_p SO(q) \otimes \pi_q SO(p) \xrightarrow{\text{PL.}} \theta_{p+q+1}$$

"plumbing"

3. Refinement of plumbing $\Rightarrow$ elements of Gromoll filtration

Proposition [ABK] PL. may be lifted to:

$$\begin{tikzcd}
\pi_p SO(q-a) \otimes \pi_q SO(p-b) \arrow[r, "\widetilde{PL.}"] \arrow[d] & \pi_{a+b+1} \mathrm{Diff}_\partial(D^{p+q-a-b-1}) \arrow[d, "\Psi"] \arrow[dd, bend left] \\
 & \pi_0 \mathrm{Diff}_\partial(D^{p+q}) \\
\pi_p SO(q) \otimes \pi_q SO(p) \arrow[r, "PL."] & \Theta_{p+q+1} \supseteq \Theta_{p+q+1}(a+b+1)
\end{tikzcd}$$

$\pi_0 \mathrm{Diff}_\partial(D^{p+q}) \cong \Theta_{p+q+1}$

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4. The subgroup $bP_{n+1} \subseteq \Theta_n$

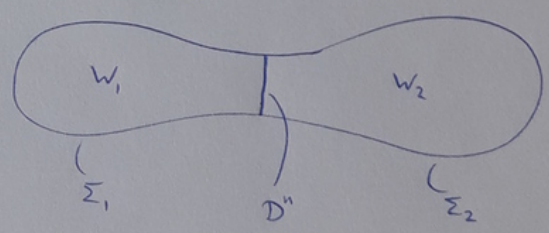
Def $bP_{n+1} = \{ \Sigma \in \Theta_n \mid \Sigma = \partial W^{n+1} \text{ with } TW \text{ trivial} \}$

Lemma This is a subgroup

$\longrightarrow$ **Idea:** $\Sigma_1 = \partial W_1$, $\Sigma_2 = \partial W_2$ $\longrightarrow$ $\Sigma_1 \# \Sigma_2 = \partial(W_1 \natural W_2)$

∂-connected sum

glue trivialisations of TW_1 and TW_2 along $W_1 \cap W_2 \cong D^n$



//.

$\underline{n = 4k-1}$.

let $\sigma_k := \gcd\left\{ |\mathrm{sign}(W)| \;:\; \begin{array}{l} W \text{ } 4k\text{-manifold} \\ TW \text{ trivial} \\ \mathrm{sign}(W) \neq 0 \end{array} \quad \partial W = S^{4k-1} \right\}$

$\tau_k := \gcd\left\{ \text{———} \;\; '' \;\; \text{———} \quad \partial W \text{ } \underline{\text{homeo}} \text{ to } S^{4k-1} \right\}$

<u>Lemma</u>. $\Sigma \in bP_{4k}$ $\qquad \Sigma = \partial W_1 = \partial W_2 \qquad (TW_i \text{ trivial})$

Then $\mathrm{sign}(W_i) \in \tau_k \mathbb{Z}$

$\mathrm{sign}(W_1) - \mathrm{sign}(W_2) \in \sigma_k \mathbb{Z}$

<u>Def</u> (Milnor)

$$\begin{array}{ccc} bP_{4k} & \xrightarrow{\lambda} & \mathbb{Z}/b_k \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \Sigma = \partial W & \longmapsto & \frac{1}{\tau_k}\,\mathrm{sign}(W) \end{array} \qquad b_k = \frac{\sigma_k}{\tau_k}$$

<u>Thm</u> (Kervaire–Milnor) This is an <u>isomorphism</u>.

$\tau_k = 8$

$\sigma_k = 2^{2k+1}\left(2^{2k-1}-1\right) \cdot a_k \cdot \mathrm{Numerator}\left(\frac{B_k}{4k}\right)$

$$a_k = \begin{cases} 1 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$$

B_k = Bernoulli number

5. Detection results (torsion)

$$\begin{tikzcd}
\pi_p SO(q-a) \otimes \pi_q SO(p-b) \arrow[r, "\widetilde{PL}."] \arrow[d, "i_*"'] & \pi_{2k-3} \mathrm{Diff}_\partial(D^{2k+1}) \arrow[d, "\Psi"] & \\
\pi_p SO(q) \otimes \pi_q SO(p) \arrow[r, "PL."] & \Theta_{4k-1} \supseteq bP_{4k} \arrow[d, "\cong", "\lambda'"'] & \\
& \mathbb{Z}/b_k &
\end{tikzcd}$$

$k = r+s$

$p = 4r-1$
$q = 4s-1$
$a = 4s-2r-2$
$b = 4r-2s-2$

[ABK]: Find elements of $\pi_p SO(q-a) \otimes \pi_q SO(p-b)$
that map to bP_{4k}
& whose λ'-invariant is $8 b_r b_s \in \mathbb{Z}/b_k$

whenever
$r, s \geq 6$
$\frac{1}{2} r < s < 2r$

<u>Construction</u>
- quite involved
- key step "pulling back" elements along i_* depends on studying the LES of $SO(n) \hookrightarrow SO(2n) \twoheadrightarrow V_{2n,n}$ in the "metastable range".

Stiefel manifold

[Barratt - Mahowald '64]

<u>Number theory</u> $\Rightarrow$ $\forall k \geq 12$ $\exists r, s \geq 6$:

$k = r+s$
$\frac{1}{2} r < s < 2r$
$8 b_r b_s \not\equiv 0 \pmod{b_k}$

<u>Cor</u>: $\forall k \geq 12$, $\boxed{\pi_{2k-3} \mathrm{Diff}_\partial(D^{2k+1}) \neq 0}$

[Burghelea - Lashof '74] find torsion in $\pi_* \mathrm{Diff}_\partial(D^d)$ of specific prime order.

E.g. they prove that $\pi_{2p-2} \mathrm{Diff}_\partial(D^{2p^2-2p-4})$ contains p-torsion (p odd prime).

6. Infinite-type results

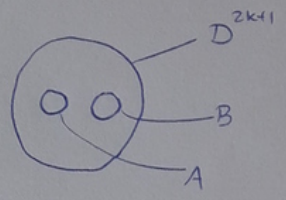
Thm [ABK] $\forall k \geq 12$

$\mathrm{Diff}_\partial(D^{2k+1})$ is not $\simeq$ <u>any</u> finite CW-complex

In fact they prove this for $\mathrm{Diff}_\partial(D^d)$, $d \geq 7$

proof: $G = \mathrm{Diff}_\partial(D^{2k+1})_0$ — identity component

claim: G is homotopy-abelian



$$\begin{tikzcd}
\mathrm{Diff}_\partial(A)_0 \times \mathrm{Diff}_\partial(B)_0 \arrow[r, hook, "\simeq"] \arrow[d, equal] & G \times G \arrow[d, equal] \arrow[rr, "\circ"] & & G \\
\mathrm{Diff}_\partial(A)_0 \times \mathrm{Diff}_\partial(B)_0 \arrow[r, hook, "\simeq"'] & G \times G \arrow[r, "\text{swap}"'] & G \times G \arrow[ur, "\circ"'] &
\end{tikzcd}$$

//.

Thm [Hubbuck '69] if G is a $\begin{pmatrix}\text{connected} \\ \text{hty-abelian}\end{pmatrix}$ H-space

& $G \simeq$ finite CW-complex

then $G \simeq (S^1)^n$.

But $(S^1)^n$ has vanishing π_*, $* \geq 2$.

whereas $\pi_{2k-3}(G) \neq 0$. $\square$

7. Watanabe's construction $\hat{Z}_2$

M homology d-sphere $\qquad \underline{\underline{d = 2k+1}}$.

$M^\circ = M \setminus \mathrm{int}(D^d)$

$$\left(\begin{array}{ll} E & \text{smooth } M^\circ\text{-bundle} \\ \downarrow \pi & \text{with trivial } \partial^{fib} \\ B & + \text{"vertical framing of } TE\text{"} \end{array}\right) \xmapsto{\quad \xi_2 \quad} H^{2k-2}(B)$$

<u>Idea</u>:

$$\begin{tikzcd} \bar{C}_2(M^\circ) \arrow[d, hook] & \supseteq & \partial \bar{C}_2(M^\circ) \arrow[r] \arrow[d, hook] & S^{2k} \\ E\bar{C}_2(\pi) \arrow[d] & \supseteq & \partial^{fib} E\bar{C}_2(\pi) \arrow[ur, "p"'] \arrow[dl] & \\ B & & & \end{tikzcd}$$

"propagator"

associated bundle whose fibre is the Fulton-MacPherson compactification of $C_2(M^\circ)$

(1) pull back gen. of $H^{2k}(S^{2k})$ along p

(2) extend (uniquely) to $\omega \in H^{2k}(E\bar{C}_2(\pi))$

(3) take <u>triple cup product</u> $\quad \omega^3 \in H^{6k}(E\bar{C}_2(\pi))$

(4) "integrate along the fibre" $\longrightarrow H^{2k-2}(B)$.

<u>Caveat</u>: Do steps (1)-(3) carefully at the level of <u>cochains</u>

$$\rightsquigarrow \text{element of } \quad H^{6k}(E\bar{C}_2(\pi), \partial^{fib} E\bar{C}_2(\pi)) \ni \tilde{\omega}^3$$

$$\downarrow$$

$$H^{6k}(E\bar{C}_2(\pi)) \ni \omega^3$$

Then step (4) works.

<u>Thm</u> (Kontsevich) This is well-defined.

<u>Def</u> (Watanabe)

$$\pi_{2k-2} B\mathrm{Diff}_\partial(D^{2k+1}) \xrightarrow{\hat{Z}_2} \mathbb{Q}$$

$$\cong$$

$$\left\{ \begin{array}{l} E \\ \pi \downarrow \\ S^{2k-2} \end{array} \quad \begin{array}{l} \text{smooth } D^{2k+1}\text{-bundle} \\ \text{with trivial } \partial^{fib} \end{array} \right\}$$

<u>Idea</u>: (1) obstructions to $\exists$ of <u>vertical framing of TE</u>

live in $\underbrace{\pi_{4k-2} SO(2k+1)}_{\text{finite}}$ [Milnor–Moore]

$q_k :=$ smallest $d \in \mathbb{Z}_{>0}$ killing all elements of this group.

Then $q_k \cdot E$ always admits a vertical framing.

(2) Choose one. $\longrightarrow$ $\zeta_2(q_k \cdot E) \in H^{2k-2}(S^{2k-2})$

$$\hat{Z}_2(E) := \underset{\mathbb{Z}}{\underset{\cap}{\langle \zeta_2(q_k \cdot E), [S^{2k-2}] \rangle}} + \underset{\mathbb{Q}}{\underset{\cap}{(\text{framing correction})}} \in \mathbb{Q}$$

<u>Thm</u> (Watanabe) This is a well-defined homomorphism.

main difficult theorem!

8. Detection results (non-torsion)

Thm (Watanabe) $k \geq 3$

$$\begin{tikzcd}
\pi_{2k-3}\mathrm{Diff}_\partial(D^{2k+1}) \arrow[d, "\Psi"'] & \cong & \pi_{2k-2}B\mathrm{Diff}_\partial(D^{2k+1}) \arrow[rr, "\hat{z}_2"] & & \mathbb{Q} & & \\
 & & \Psi^{-1}(bP_{4k}) \arrow[u, phantom, "\cup"] \arrow[d] \arrow[rr] & & \frac{1}{b_k}\mathbb{Z} \arrow[u, phantom, "\cup"] \arrow[r, "\cdot b_k"] & \mathbb{Z} \arrow[d, two heads] & \\
\Theta_{4k-1} & \supseteq & bP_{4k} \arrow[r, "\lambda'", "\cong"'] & \mathbb{Z}/b_k \arrow[rr, "\cdot c_k q_k"] & & \mathbb{Z}/p_k &
\end{tikzcd}$$

$\mathbb{Z}/b_k \ni 8b_rb_s$

$b_k, c_k \longrightarrow$ Bernoulli #s

$q_k \longleftrightarrow$ torsion in $\pi_{4k-2}SO(2k+1)$

Recall that [A&K] $\longrightarrow$ $x \in \Psi^{-1}(bP_{4k})$

with $\lambda'(\Psi(x)) = 8b_rb_s \in \mathbb{Z}/b_k$

$k = r+s$
$r, s \geq 6$
$\frac{1}{2}r < s < 2r$

Cor If $8b_rb_sc_kq_k \not\equiv 0 \pmod{b_k}$ $(*)$

then $\pi_{2k-3}\mathrm{Diff}_\partial(D^{2k+1})$ contains an element of ∞ order.

i.e. $\pi_{2k-3}\mathrm{Diff}_\partial(D^{2k+1}) \otimes \mathbb{Q} \neq 0$.

Watanabe checks $(*)$ for $k = 4, 5, 12, \ldots, 41, 45, 54, \ldots, 2212$.

also any $k = 2m$ where $4m-1$ is prime & $8m-1$ is a regular prime.

Note:
- In the stable range, non-zero $\pi_*(-)\otimes\mathbb{Q}$ occur only for $* \equiv 3 \pmod 4$.
- Whenever $(*)$ holds for even k, we get non-zero $\pi_*(-)\otimes\mathbb{Q}$ for $* \equiv 1 \pmod 4$.