On homological stability for configuration-section spaces

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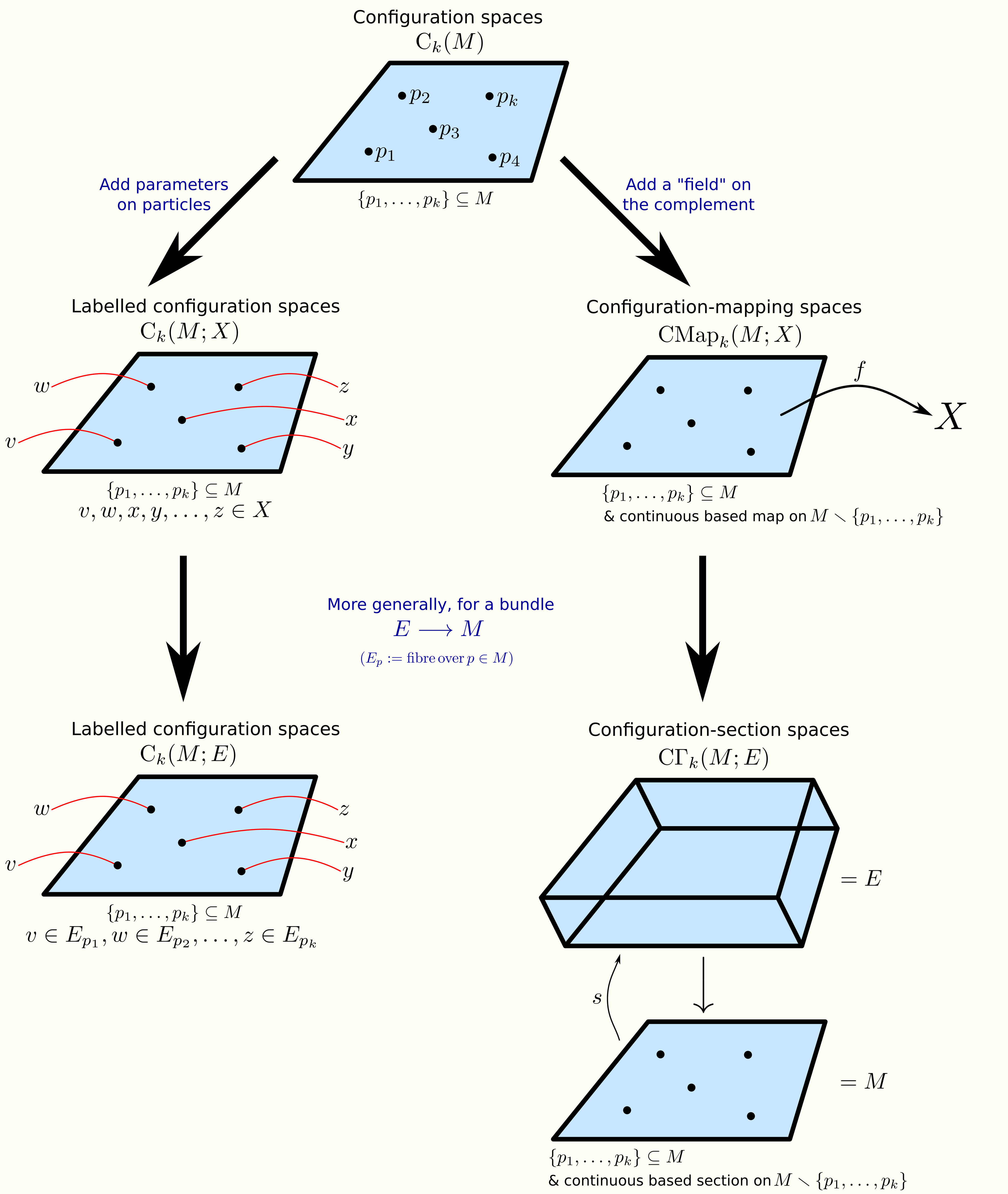
Configuration-section spaces

For a connected manifold M of dimension $d \geqslant 2$ with basepoint $* \in \partial M$:



Examples.

- *Braid groups* $C_k(\mathbb{R}^2) \simeq K(B_k,1)$ and *ribbon braid groups* $C_k(\mathbb{R}^2;S^1) \simeq K(rB_k,1)$.
- For a group G, the *Hurwitz spaces* $\mathrm{CMap}_k(\mathbb{R}^2;K(G,1)) \simeq \mathrm{Hur}_{G,k}$. These spaces parametrise pairs $(\nu\colon \Sigma \to \mathbb{R}^2, G \hookrightarrow \mathrm{Deck}(\nu))$ where ν is a covering with k branch points and G acts transitively on its generic fibres. Fixing the *charge* (see next column) corresponds to restricting the *monodromy* of the covering around the branch points.
- When M is smooth and $E = TM \smallsetminus \{\text{zero-section}\}$, the space $C\Gamma_k(M;E)$ parametrises configurations together with a non-vanishing vector field on their complement. Fixing the *charge* (see next column) corresponds to prescribing the *winding number* of this vector field around each configuration point (up to sign, if M is non-orientable).

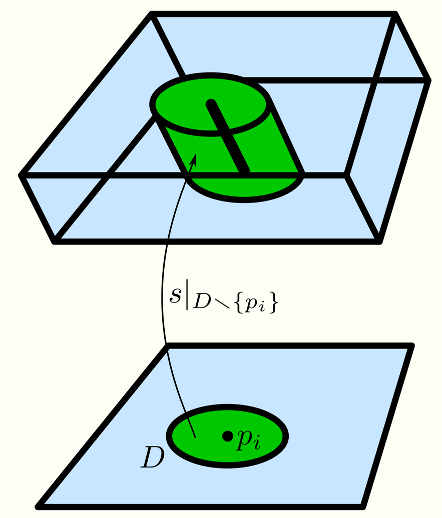
Charges

For $(\{p_1,\ldots,p_k\},s) \in C\Gamma_k(M;E)$ we interpret:

- $p_1,\ldots,p_k$ as *particles* in the ambient space M;
- s as a *field* defined away from the particles;
- the *charge* of the field s at p_i is:

$$s|_{D \smallsetminus \{p_i\}} \qquad \text{(up to homotopy)},$$

where $D \subset M$ is a codimension-zero disc centred at p_i. Pictorially:



For trivial bundles $E = M \times X$ and orientable M, the *set of possible charges* is $[S^{d-1},X] = \{$homotopy classes of maps $S^{d-1} \to X\}$.

Eg. For Hurwitz spaces, $\{$*charges*$\} = [S^1, BG] \leftrightarrow \{$conj. classes of $G\}$.

In general, it is a quotient of the set $[S^{d-1},X]$. We have surjections:

$$\{\textit{charges}\} \twoheadleftarrow [S^{d-1},X] \twoheadleftarrow \pi_{d-1}(X).$$

Def. The subspace of $C\Gamma_k(M;E)$ where all charges are equal to c is:

$$C\Gamma_k^c(M;E)$$

Def. A charge c is *small* if its pre-image in $\pi_{d-1}(X)$ has size 1.

Main theorem — homological stability

Theorem [PT₁]. If the *charge* c is small, the integral homology groups

$$H_i(C\Gamma_k^c(M;E);\mathbb{Z})$$

are independent of the number of particles k when $k \geqslant 2i+4$.

Related results and examples

- Homological stability for configuration spaces $C_k(M)$ is a classical result [S_1,M,S_2], and generalises to labelled configuration spaces [RW].
- Recently, [EVW] have proven homological stability for Hurwitz spaces

$$\{\mathrm{Hur}_{G,k}^c\}_{k=1}^{\infty} \tag{1}$$

(with a certain periodicity and with coefficients in a field containing $|G|^{-1}$) if G is finite, c generates G and c does not split into multiple conjugacy classes when restricted to subgroups of G. Using this, they proved an asymptotic version of the *Cohen-Lenstra conjecture*. A special case of our main theorem gives a complementary result — the sequence of Hurwitz spaces (1) is homologically stable if c is a conjugacy class of size 1.
- Assume M is smooth and fix $w \in \mathbb{Z}$ ($w = 0$ if M is non-orientable). Denote by $V_k^w(M)$ the moduli space of non-vanishing vector fields defined on M except at k "singularities", with winding number w at each singularity. Then $\{V_k^w(M)\}_{k=0}^{\infty}$ is homologically stable.

Outline of proof

There are maps f that forget the field and s that increase the number of particles by one (adding a new particle and extending the field near the basepoint $* \in \partial M$):

$$\begin{array}{ccc}
\Gamma^c(M \smallsetminus \{k \text{ points}\};E) & \xrightarrow{\;s\;} & \Gamma^c(M \smallsetminus \{k+1 \text{ points}\};E) \\
\downarrow & & \downarrow \\
C\Gamma_k^c(M;E) & \xrightarrow{\;s\;} & C\Gamma_{k+1}^c(M;E) \\
\downarrow f & & \downarrow f \\
C_k(\mathring{M}) & \xrightarrow{\;s\;} & C_{k+1}(\mathring{M})
\end{array} \tag{2}$$

[1] By a Serre spectral sequence argument it suffices to show that

$$H_i\Big(C_k(\mathring{M})\;;\;H_j(\Gamma^c(M \smallsetminus \{k \text{ points}\};E))\Big) \tag{3}$$

stabilises (with respect to k) for each fixed $j \geqslant 0$.

[2] For fixed j, the coefficient system

$$k \longmapsto H_j(\Gamma^c(M \smallsetminus \{k \text{ points}\};E))$$

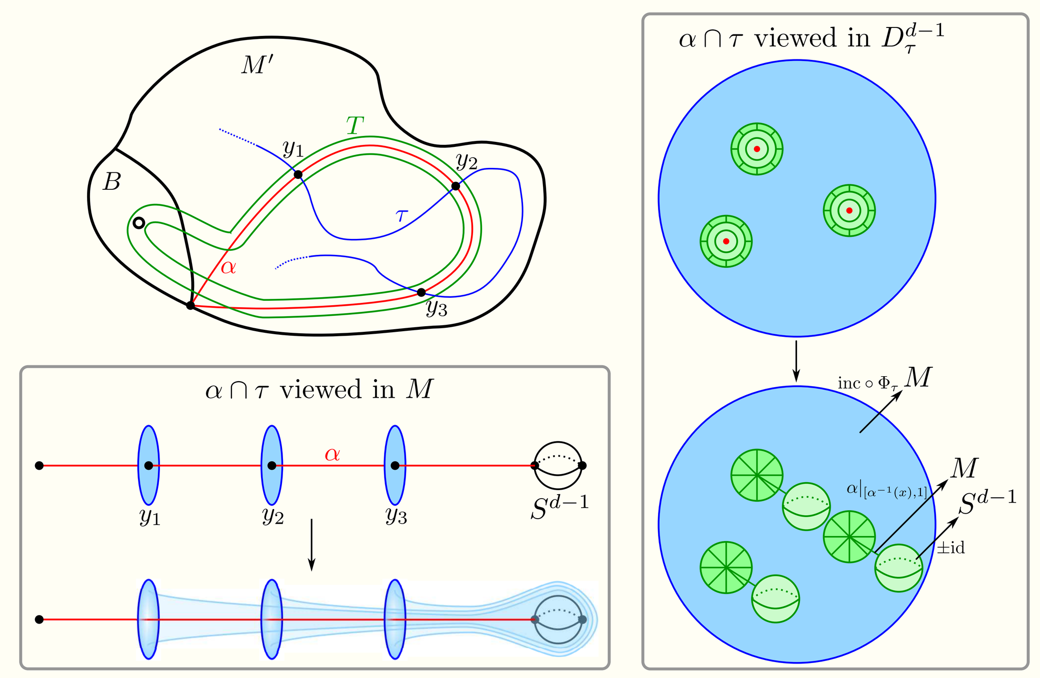
forms a *polynomial* twisted coefficient system for the configuration spaces $C_k(\mathring{M})$. *The proof of this fact uses crucially the assumption that the charge c is small.*

[3] It is known [P,K] that configuration spaces are homologically stable with respect to *polynomial* twisted coefficient systems. Thus (3) stabilises for each fixed j.

Further results

The fibration sequence on the left of (2) has an associated *monodromy action* (up to homotopy) of $\pi_1(C_k(\mathring{M}))$ on the fibre $\Gamma^c(M \smallsetminus \{k \text{ points}\};E)$. Now assume that $d \geqslant 3$ and $E = M \times X$. The group $\pi_1(C_k(\mathring{M}))$ decomposes as $\pi_1(M)^n \rtimes \mathfrak{S}_n$ and the fibre decomposes as $\mathrm{Map}_*(M,X) \times (\Omega_c^{d-1}X)^k$.

Theorem [PT₂]. An explicit description of the monodromy action under these identifications. If M is simply-connected or has handle-dimension $\leqslant \dim(M) - 2$, this is a direct formula; otherwise, it is more subtle to describe.



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mpanghel@imar.ro