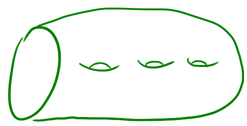
Mapping class group representations via Heisenberg and Schrödinger

Joint with Christian Blanchet
Awais Shaukat

ArXiv: 2109.00515

IMAR
Topology Seminar
12 November 2021



Goal: Construct interesting representations of MCGs of surfaces Σ.

homological

$$\mathcal{M}(\Sigma) = \pi_0 \operatorname{Diff}_\partial(\Sigma)$$

Motivation: (why representations?)

Understand their structure through the lens of symmetries of some simpler objects
- vector spaces
- modules over "nice" rings
  - Laurent polynomials $\mathbb{Z}[\mathbb{Z}^k]$
  - $\mathbb{Z}[G]$ for groups G that are close to abelian
  - ~~$\mathbb{Z}[\mathcal{M}(\Sigma)]$~~

(why homological?)

Mapping class groups arise topologically
$\Rightarrow$ should be understood topologically

First examples

(1) $\boxed{\Sigma = D_n = \mathbb{D}^2 \setminus \{n \text{ points}\}}$

$$\mathcal{M}(\Sigma) = \underbrace{B_n}_{\text{Artin braid group}} \curvearrowright H_1(\mathbb{D}_n) \cong \mathbb{Z}^n$$

This is just $B_n \twoheadrightarrow \mathfrak{S}_n \hookrightarrow GL_n(\mathbb{Z})$

(2) Buran representation ('30s)

$$\mathcal{M}(\Sigma) = B_n \curvearrowright H_1(\widetilde{\mathbb{D}}_n) \cong \mathbb{Z}[\mathbb{Z}]^{n-1}$$

$\cong H_1(\mathbb{D}_n; \mathbb{Z}[\mathbb{Z}])$

Shapiro's lemma

$$\begin{array}{c} \widetilde{\mathbb{D}}_n \\ \downarrow \\ \mathbb{D}_n \end{array} \qquad \begin{array}{c} \pi_1(\mathbb{D}_n) \\ \Big\downarrow \text{total winding number} \\ \mathbb{Z} \end{array}$$

B_n-invariant

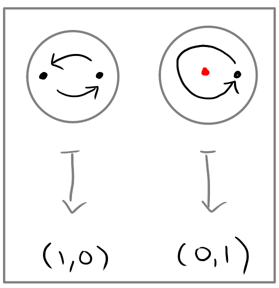
(3) Lawrence representations ('90s)
$k \geq 2$

$$\mathcal{M}(\Sigma) = B_n \curvearrowright H_k(C_k(\mathbb{D}_n); \mathbb{Z}[\mathbb{Z}^2])$$

free module of rank $\binom{k+n-2}{k}$ over $\mathbb{Z}[\mathbb{Z}^2]$

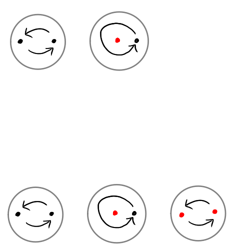
$$\begin{array}{c} \widetilde{C_k(\mathbb{D}_n)} \\ \downarrow \\ C_k(\mathbb{D}_n) \end{array} \qquad \begin{array}{c} \pi_1(C_k(\mathbb{D}_n)) \\ \downarrow \\ \mathbb{Z}^2 \end{array}$$

B_n-invariant



Alternative construction:

$$\begin{array}{ccc} \pi_1 C_k(\mathbb{D}_n) & \longrightarrow\!\!\!\!\rightarrow & \mathbb{Z}^2 \\ \big\downarrow & & \big\downarrow \\ B_{k,n} & \xrightarrow{\ ab\ }\!\!\!\!\rightarrow & \mathbb{Z}^3 \end{array}$$



Burau $(k=1)$ $\quad B_n \longrightarrow GL_{n-1}(\mathbb{Z}[\mathbb{Z}]) \hookrightarrow GL_{n-1}(\mathbb{R})$ $\quad$ choice of transcendental number

Lawrence $(k \geq 2)$ $\quad B_n \longrightarrow GL_{\binom{k+n-2}{k}}(\mathbb{Z}[\mathbb{Z}^2]) \hookrightarrow GL_{\binom{k+n-2}{k}}(\mathbb{R})$ $\quad$ choice of two algebraically independent transcendental numbers

<u>Theorem</u> (Bigelow, Krammer '00)

The $k=2$ Lawrence representation $B_n \longrightarrow GL_{\binom{n}{2}}(\mathbb{Z}[\mathbb{Z}^2])$ is <u>faithful</u>

injective

Hence B_n is a <u>linear</u> group for all n.

Lawrence representations (for all k) are also used in constructing <u>topological models of quantum link invariants</u>. [Kohno, Ito, Anghel, Martel]

<u>General recipe</u> (first version)

$$\pi_1 C_k(\Sigma) \overset{\phi}{\twoheadrightarrow} Q \quad \mathcal{M}(\Sigma)\text{-invariant}$$

$$\leadsto \quad \widetilde{C_k(\Sigma)}^{\phi} \longrightarrow C_k(\Sigma)$$

regular covering with Deck $\cong Q$

$\mathcal{M}(\Sigma)$-action lifts to $\widetilde{C_k(\Sigma)}^{\phi}$ and commutes with Q-action

$$\leadsto \quad \mathbb{Z}\left[\widetilde{C_k(\Sigma)}^{\phi}\right] \otimes_{\mathbb{Z}[Q]} V \longrightarrow C_k(\Sigma)$$

Q-representation over R

bundle of R-modules

local system for (co)homology

$\mathcal{M}(\Sigma)$ acts on $C_k(\Sigma)$ <u>together with</u> this local system

twisted homology

$$\leadsto \quad H_i(C_k(\Sigma); V)$$

$\mathcal{M}(\Sigma)$-representation over R.

$\pi_1 C_k(\Sigma)$

$\downarrow \phi$ $\mathcal{M}(\Sigma)$-invariant

Q

V Q-representation over R

$\rightsquigarrow \boxed{} \rightsquigarrow$ $\mathcal{M}(\Sigma)$-representation over R

Studied for MCGs and motion groups in all dimensions, and made "functorial", in [P.-Soulié, arxiv: 1910.13423]

Flavours

In the final step, instead of $H_i : (\text{spaces} + \text{local systems}) \longrightarrow R\text{-modules}$

we could apply any of the functors

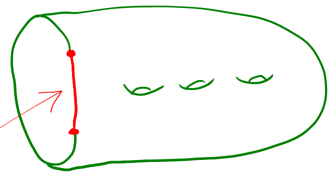
- $H_i(-,-) : (\text{pairs of spaces} + \text{local systems}) \longrightarrow R\text{-modules}$ Relative homology
- $H_i^{BM} : \left(\begin{array}{l}\text{loc. cpt spaces}\\ \text{proper maps}\end{array} + \text{local systems}\right) \longrightarrow R\text{-modules}$ Borel-Moore homology ("locally finite")
- ...

The relations between these different flavours, for fixed inputs
$\begin{cases} \phi : \pi_1 C_k(\Sigma) \twoheadrightarrow Q \\ Q\text{-representation } V, \end{cases}$
are studied in [Anghel-P., arxiv: 2011.02388]

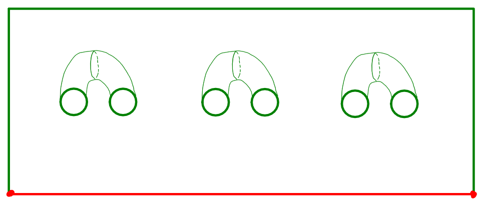
We will use the following version:

$$H_k^{BM}(C_k(\Sigma), \partial_0 C_k(\Sigma); V)$$

where $\partial_0 C_k(\Sigma)$ = subspace of configurations in which at least one point lies in $\partial_0 \Sigma$



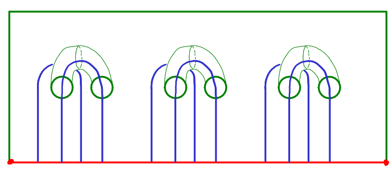
$\cong$



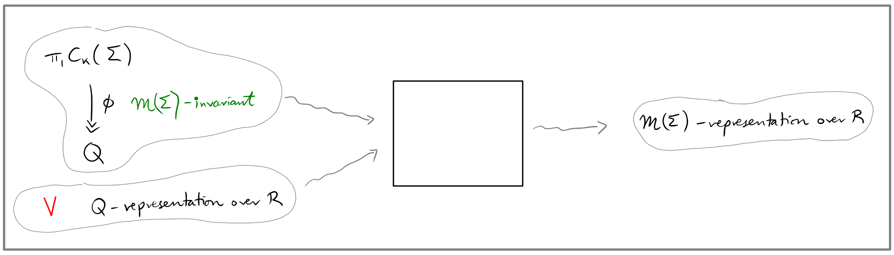
Proposition:

$$H_k^{BM}(C_k(\Sigma), \partial_0 C_k(\Sigma); V) \cong \bigoplus V$$

partitions of k into $2g$ non-negative integers



However, the machine



is limited, since there are interesting quotients $\pi_1 C_k(\Sigma) \xrightarrow{\phi} Q$ that are not $\mathcal{M}(\Sigma)$-invariant (although $\ker(\phi)$ is $\mathcal{M}(\Sigma)$-invariant)

Theorem (Blanchet–P.–Shaukat, arxiv: 2109.00515) $\boxed{\Sigma = \Sigma_{g,1} = }$

Associated to any representation V of the Heisenberg group $\mathcal{H} = \mathcal{H}(\Sigma)$, we construct a family (indexed by integers $k \geq 2$) of

1. twisted representations of $\mathcal{M}(\Sigma)$
2. representations of $\mathrm{Torelli}(\Sigma)$
3. representations of $\widetilde{\mathcal{M}}(\Sigma)$, the universal central extension of $\mathcal{M}(\Sigma)$
4. representations of $\mathrm{Morita}(\Sigma)$

when $V =$ Schrödinger repr. of $\mathcal{H}$ (for 3 and 4)

Heisenberg group

- Central extension $1 \to \mathbb{Z} \to \mathcal{H}(\Sigma) \to H_1(\Sigma) \to 1$ (2-nilpotent)

  corresponding to the intersection cocycle $H_1(\Sigma) \times H_1(\Sigma) \xrightarrow{\cdot} \mathbb{Z}$

- Concretely $\mathcal{H}(\Sigma) = \mathbb{Z} \times H_1(\Sigma)$

$$(a,x)(b,y) = (a+b+x\cdot y\,,\ x+y)$$

Torelli(Σ) $= \ker\left(\mathcal{M}(\Sigma) \xrightarrow{\text{action on } H_1} \mathrm{Aut}(H_1(\Sigma))\right)$

Morita(Σ) $= \ker\left(\mathcal{M}(\Sigma) \xrightarrow{\partial} H_1(\Sigma)\right)$

crossed homomorphism, defined by Morita ('85) (combinatorial definition)

reinterpreted geometrically by Trapp ('92) in terms of winding numbers & non-vanishing vector fields

we will give a third interpretation later...

universal central extension of a group G

- Admits a unique morphism (of central ext) to any other central extension of G

  (initial object in the category of central extensions of G)

- Exists if and only if $H_1(G) = 0$
- When it exists, it is an extension of G by $H_2(G)$ "Schur multiplier of G"

When $G = \mathcal{M}(\Sigma_{g,1})$

- $H_1(G) = 0$ for $g \geq 3$
- $H_2(G) \cong \mathbb{Z}$ for $g \geq 4$

Smallprint

For $g \geq 4$ we consider the universal central extension For $g \leq 3$ we consider the pullback of the u.c.e. of $\mathcal{M}(\Sigma_{4,1})$ along the stabilisation map $\mathcal{M}(\Sigma_{g,1}) \to \mathcal{M}(\Sigma_{4,1})$

twisted representations

A representation of a group G is a functor $G \longrightarrow Mod_R$

one-object category

category of R-modules

A twisted representation of G is a functor $\underbrace{Ac(G \curvearrowright X)}$ $\longrightarrow Mod_R$

groupoid that "fragments" G into many objects according to a given action of G on a set X } "action groupoid"

E.g. $X = *$ $\quad Ac(G \curvearrowright *) = G$

$X = G$ $\quad Ac(G \curvearrowright G) =$ "translation groupoid"

- obj $= G$
- exactly one morphism between every pair of objects

Schrödinger representation of $\mathcal{H}(\Sigma)$

— A certain irreducible representation of $\mathcal{H}(\Sigma)$ by unitary operators on $L^2(\mathbb{R}^g)$.

(More details later...)

Non-triviality

Prop (corollary of [Moriyama '07])

All of our (twisted) representations have kernel $\subseteq \underbrace{J(k)}_{\text{Johnson filtration of } \mathcal{M}(\Sigma)} \cap \overbrace{\text{Magnus}}^{\text{Originally defined using Fox calculus}}$

$$\bigcap_k J(k) = \{id\}$$

Diagram

$$\begin{tikzcd}
 & \mathrm{Torelli}(\Sigma) \arrow[d, phantom, "\shortparallel"] & & & & \\
\mathcal{M}(\Sigma) & J(1) \arrow[l, hook'] & J(2) \arrow[l, hook'] \arrow[dl] & J(3) \arrow[l, hook'] & J(4) \arrow[l, hook'] & \cdots \arrow[l, hook'] \\
 & \mathrm{Morita}(\Sigma) \arrow[ul, hook] & & & &
\end{tikzcd}$$

Calculation

$\Sigma = \Sigma_{1,1} =$ 

$k=2$

$V = \mathbb{Z}[\mathcal{H}]$ regular representation of $\mathcal{H}$ over $R = \mathbb{Z}[\mathcal{H}] = \mathbb{Z}[u^{\pm 1}]\langle a^{\pm 1}, b^{\pm 1}\rangle \big/ ([a,b] = u^2)$

In this case the representation $H_2^{BM}(C_2(\Sigma), \partial_0 C_2(\Sigma); \mathbb{Z}[\mathcal{H}])$ is 3-dimensional over $\mathbb{Z}[\mathcal{H}]$, and $T_{\partial\Sigma}$ acts by the matrix ...

Dehn twist around the boundary ($\in J(2)$)

$$\begin{pmatrix}
\begin{matrix} u^{-8}b^2+u^{-4}a^{-2}-ua^{-2}b^2+(u^{-1}-u^{-2})a^{-2}b+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(u^{-4}-u^{-5})a^{-1}b \end{matrix} &
\begin{matrix} (u^2+1-2u^{-1}+u^{-2}+u^{-4})a^{-2}b^2-ua^{-2}b^4+ \\ (-u^2+u+u^{-1}-u^{-2})a^{-2}b^3-u^{-3}a^{-2}+ \\ (-1+u^{-1}+u^{-3}-u^{-4})a^{-2}b \end{matrix} &
\begin{matrix} (-1+2u^{-1}-u^{-2}-u^{-4}+u^{-5})a^{-2}b+ \\ (u-1)a^{-2}b^3+(u^2-u-u^{-1}+2u^{-2}-u^{-3})a^{-2}b^2+ \\ (-u^{-3}+u^{-4})a^{-1}b+(u^{-4}-u^{-5})a^{-1}b^3+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}b^2+ \\ (-u^{-3}+u^{-4})a^{-2} \end{matrix} \\
\begin{matrix} -u^{-1}-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-2}a^2+ \\ (u^{-1}-u^{-2}-u^{-4}+u^{-5})a+u^{-6}a^{-2}+ \\ (u^{-3}-u^{-4}-u^{-6}+u^{-7})a^{-1} \end{matrix} &
\begin{matrix} 1+u^{-2}-u^{-3}+u^{-6}+u^{-6}a^{-2}b^2-u^{-1}b^2+ \\ (u^{-3}-u^{-4})a^{-1}b^2+(-1+u^{-1}+u^{-3}-u^{-4})b+ \\ (u^{-2}-2u^{-3}+u^{-4}+u^{-6}-u^{-7})a^{-1}b-u^{-5}a^{-2}+ \\ (-u^{-2}+u^{-3}+u^{-5}-u^{-6})a^{-1}+(u^{-5}-u^{-6})a^{-2}b \end{matrix} &
\begin{matrix} (-u^{-6}+u^{-7})a^{-2}b+ \\ (u^{-1}-u^{-2}-u^{-4}+2u^{-5}-u^{-6})b+ \\ (-u^{-3}+2u^{-4}-u^{-5}-u^{-7}+u^{-8})a^{-1}b+ \\ 1-u^{-1}+u^{-2}-3u^{-3}+2u^{-4}+u^{-6}-u^{-7}+ \\ (-u^{-2}+2u^{-3}-u^{-4}+u^{-5}-2u^{-6}+u^{-7})a^{-1} \\ +(u^{-2}-u^{-3})ab+(-1+u^{-1}+u^{-3}-u^{-4})a+ \\ (-u^{-5}+u^{-6})a^{-2} \end{matrix} \\
\begin{matrix} -u^{-6}ab+(-u^{-3}+u^{-4}-u^{-7})b-u^{-4}+ \\ (u^{-1}-u^{-4}+u^{-5})a^{-1}b+u^{-2}a^{-2}b+ \\ (-u^{-3}+u^{-6})a^{-1}+u^{-5}a^{-2} \end{matrix} &
\begin{matrix} (-1-u^{-2}+2u^{-3}-u^{-6})a^{-1}b+u^{-1}a^{-1}b^3+ \\ u^{-2}a^{-2}b^3+(1-u^{-1}-u^{-3}+u^{-4})a^{-1}b^2+ \\ (u^{-1}-u^{-2}+u^{-5})a^{-2}b^2+(-u^{-1}+u^{-4}-u^{-5})a^{-2}b+ \\ (u^{-2}-u^{-5})a^{-1}-u^{-4}a^{-2} \end{matrix} &
\begin{matrix} u^{-3}+(u^{-2}-u^{-3}-u^{-5}+u^{-6})a^{-1}+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-1}b^2+ \\ (-u^{-2}+u^{-3})a^{-2}b^2+ \\ (-1+u^{-1}+2u^{-3}-3u^{-4}+u^{-7})a^{-1}b+ \\ (-u^{-1}+u^{-2}-u^{-5}+u^{-6})a^{-2}b+(-u^{-4}+u^{-5})b^2+ \\ (u^{-2}-u^{-3}-u^{-5}+u^{-6})b+(-u^{-4}+u^{-5})a^{-2} \end{matrix}
\end{pmatrix}$$

$\boxed{(1)}$ Idea of construction of representations of $\mathrm{Torelli}(\Sigma)$ (V any representation of $\mathcal{H}(\Sigma)$)

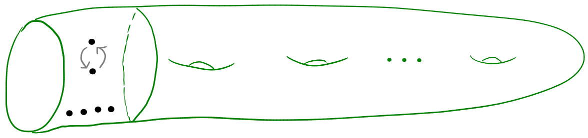
$\boxed{(2)}$ Idea of construction of representations of $\widetilde{\mathcal{M}}(\Sigma)$ (V = Schrödinger representation)

$\boxed{(1)}$

Step 1 $\exists$ projection $\pi_1 C_K(\Sigma) \xrightarrow{\phi} \mathcal{H}(\Sigma)$ whose kernel is $\mathcal{M}(\Sigma)$-invariant

(hence there is a (non-trivial) induced action $\mathcal{M}(\Sigma) \curvearrowright \mathcal{H}(\Sigma)$)

Idea of proof

$\sigma_1 :=$  $\in \pi_1 C_K(\Sigma)$

Then $\pi_1 C_K(\Sigma) \big/ \langle\langle \{ [x,\sigma_1] \mid x \in \pi_1 C_K(\Sigma) \} \rangle\rangle \cong \mathcal{H}(\Sigma)$

using a presentation of $\pi_1 C_K(\Sigma)$ due to [Bellingeri]

$\ker(\phi) = \langle\langle \{ [x,\sigma_1] \mid x \in \pi_1 C_K(\Sigma) \} \rangle\rangle$

σ_1 is fixed by $\mathcal{M}(\Sigma)$ since it is supported in a collar neighbourhood of $\partial\Sigma$

Step 2 Denote the induced action on $\mathcal{H}(\Sigma)$ by $\mathcal{M}(\Sigma) \xrightarrow{\Phi} \mathrm{Aut}^+(\mathcal{H}(\Sigma))$

Then
- $\Phi^{-1}(\mathrm{Inn}(\mathcal{H}(\Sigma))) = \mathrm{Torelli}(\Sigma)$
- $\ker(\Phi) = \mathrm{Torelli}(\Sigma) \cap \mathrm{Morita}(\Sigma)$

Idea

$$\begin{array}{ccc} \mathrm{Aut}^+(\mathcal{H}(\Sigma)) & \cong & Sp(H) \ltimes H \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ \mathrm{Inn}(\mathcal{H}(\Sigma)) & \cong & \{id\} \ltimes 2H \end{array}$$

(general fact about automorphism groups of 2-nilpotent groups)

$H = H_1(\Sigma)$

$$\begin{tikzcd} & Sp(H) \\ \mathcal{M}(\Sigma) \arrow[ur, "\text{action on } H_1"] \arrow[r, "\Phi"] \arrow[dr, "\partial"'] & Sp(H) \ltimes H \arrow[u, two heads] \arrow[d] \\ & H \end{tikzcd}$$

crossed hom.

Morita's crossed hom.

Step 3

$$\begin{tikzcd} \mathbb{Z} \arrow[r, equal] \arrow[d, hook] & \text{centre} \cong \mathbb{Z} \arrow[d, hook] \\ \widetilde{\mathrm{Torelli}}(\Sigma) \arrow[r, "\widetilde{\Phi}"] \arrow[d, two heads] \arrow[dr, phantom, "\text{pullback}"] & \mathcal{H}(\Sigma) \arrow[d, two heads] \\ \mathrm{Torelli}(\Sigma) \arrow[r, "\Phi"'] & \mathrm{Inn}(\mathcal{H}(\Sigma)) \cong H_1(\Sigma) \end{tikzcd}$$

$$\begin{tikzcd} \pi_1 C_k(\Sigma) \arrow[d, two heads, "\phi"] \\ \mathcal{H}(\Sigma) \end{tikzcd} \qquad \longrightarrow \qquad \begin{tikzcd} \widetilde{C_k(\Sigma)}^{\phi} \arrow[d] \\ C_k(\Sigma) \end{tikzcd} \qquad \longrightarrow \qquad \begin{tikzcd} \mathbb{Z}\left[\widetilde{C_k(\Sigma)}^{\phi}\right] \arrow[d] \\ C_k(\Sigma) \end{tikzcd}$$

$\ker(\phi)$ is $\mathcal{M}(\Sigma)$-invt.
(but ϕ itself is not)

$\mathcal{M}(\Sigma)$-action lifts to the covering, but does not commute with the action of Deck $= \mathcal{H}(\Sigma)$

$\mathcal{M}(\Sigma)$-action doesn't preserve the local system

Restricting to $\widetilde{\mathrm{Torelli}}(\Sigma) \twoheadrightarrow \mathrm{Torelli}(\Sigma) \leqslant \mathcal{M}(\Sigma)$, we may use $\widetilde{\Phi}$ to "untwist" the action on $\mathbb{Z}\left[\widetilde{C_k(\Sigma)}^{\phi}\right]$ $\longrightarrow$

$$\begin{tikzcd} \mathbb{Z}\left[\widetilde{C_k(\Sigma)}^{\phi}\right] \otimes_{\mathbb{Z}[\mathcal{H}(\Sigma)]} V \arrow[d] \\ C_k(\Sigma) \end{tikzcd}$$

action of $\widetilde{\mathrm{Torelli}}(\Sigma)$ on $C_k(\Sigma)$ together with the local system

$\xrightarrow{H_k^{BM}(-,\partial_0)}$ Representation of $\widetilde{\mathrm{Torelli}}(\Sigma)$

Step 4 $\boxed{\widetilde{\mathrm{Torelli}}(\Sigma) \twoheadrightarrow \mathrm{Torelli}(\Sigma) \text{ is a } \underline{\text{trivial}} \text{ central extension!}}$

$\longrightarrow$ Compose with a section to obtain a representation of $\mathrm{Torelli}(\Sigma)$

- $\mathrm{Torelli}(\Sigma) \hookrightarrow \mathcal{M}(\Sigma)$ induces 0-map on $H^2(-;\mathbb{Z})$
- $H^2(-;\mathbb{Z})$ classifies central extensions by $\mathbb{Z}$
- Hence any $\mathbb{Z}$-central extension of $\mathrm{Torelli}(\Sigma)$ that extends to $\mathcal{M}(\Sigma)$ must be trivial
- $\widetilde{\mathrm{Torelli}}(\Sigma) \twoheadrightarrow \mathrm{Torelli}(\Sigma)$ is represented by the cocycle $\langle f,g\rangle = \partial(f).\partial(g)$
- [Morita] $\exists$ cocycle on $\mathcal{M}(\Sigma)$ defined by $\langle f,g\rangle = \partial(f).f_*(\partial(g))$

$\boxed{2}$ Idea of construction of representations of $\widetilde{\mathcal{M}}(\Sigma)$

Schrödinger representation of $\mathcal{H}(\Sigma)$

$$\mathcal{H}(\Sigma) = \mathbb{Z}\text{-central extension of } H_1(\Sigma) = \mathbb{Z}^{2g}$$

$$= \underset{b_1,\dots,b_g}{\mathbb{Z}^g} \ltimes \underset{a_1,\dots,a_g,u}{\mathbb{Z}^{g+1}} \qquad b_i : \begin{cases} a_i \longmapsto a_i + 2u \\ a_j \longmapsto a_j \quad j \neq i \end{cases}$$

discrete Heisenberg group

$$\downarrow$$

$$\mathcal{H}(\Sigma)_c = \underset{b_1,\dots,b_g}{\mathbb{R}^g} \ltimes \underset{a_1,\dots,a_g,u}{\mathbb{R}^{g+1}} \qquad b_i : \begin{cases} a_i \longmapsto a_i + 2u \\ a_j \longmapsto a_j \quad j \neq i \end{cases}$$

continuous Heisenberg group

1-dimensional unitary representation

$$\underset{a_1,\dots,a_g,u}{\mathbb{R}^{g+1}} \twoheadrightarrow \underset{u}{\mathbb{R}} \longrightarrow S^1 = U(1)$$

$$t \longmapsto e^{\hbar i t/2} \qquad \hbar > 0 \text{ fixed} \quad (\text{Planck's constant})$$

This induces a unitary representation on $\mathcal{H}(\Sigma)_c \gtrdot \mathbb{R}^{g+1}$

$$\mathcal{H}(\Sigma)_c \xrightarrow{\ W\ } U(L^2(\mathbb{R}^g))$$

Formula: $W(q_1 b_1 + \cdots + q_g b_g + p_1 a_1 + \cdots + p_g a_g + tu)(\psi)(s) = e^{i\hbar(p\cdot s + \frac{1}{2}(k - p\cdot q))}\psi(s-q)$

Stone - von Neumann theorem $\Longrightarrow$ $\forall \varphi \in \mathrm{Aut}(\mathcal{H}(\Sigma)_c)$,

$\exists$ unique inner automorphism $T(\varphi)$ of $U(L^2(\mathbb{R}^g))$ such that

$$\begin{array}{ccc} \mathcal{H}(\Sigma)_c & \xrightarrow{\ W\ } & U(L^2(\mathbb{R}^g)) \\ \varphi \downarrow & & \downarrow T(\varphi) \\ \mathcal{H}(\Sigma)_c & \xrightarrow{\ W\ } & U(L^2(\mathbb{R}^g)) \end{array}$$

commutes.

$$\rightsquigarrow \quad \mathrm{Aut}(\mathcal{H}(\Sigma)_c) \xrightarrow{\ T\ } \mathrm{Inn}(U(L^2(\mathbb{R}^g))) = PU(L^2(\mathbb{R}^g))$$

"Segal - Shale - Weil projective representation"

$\exists$ by universality

central ext by S^1

central extension by S^1

$$\begin{tikzcd}
\widetilde{\mathcal{M}}(\Sigma) \arrow[r] \arrow[d] & \overline{\mathcal{M}}(\Sigma) \arrow[rrr] \arrow[d, dash] & & & U(L) \arrow[d] \\
\mathcal{M}(\Sigma) \arrow[r, "\Phi"] & \mathrm{Aut}(\mathcal{H}(\Sigma)) \arrow[r, hook] & \mathrm{Aut}(\mathcal{H}(\Sigma)_c) \arrow[rr, "T"] & & PU(L)
\end{tikzcd}$$

universal central extension (by $\mathbb{Z}$)

pullback

$L = L^2(\mathbb{R}^g)$

As before, the lift $\widetilde{\mathcal{M}}(\Sigma) \longrightarrow U(L)$

of $\mathcal{M}(\Sigma) \longrightarrow PU(L)$

allows one to "untwist" the action of $\widetilde{\mathcal{M}}(\Sigma)$ on the local system

$$\begin{tikzcd}
\pi_1 C_n(\Sigma) \arrow[d, two heads, "\phi"] & \mathbb{Z}\left[\widetilde{C_k(\Sigma)}^{\phi}\right] \underset{\mathbb{Z}[\mathcal{H}(\Sigma)]}{\otimes} L^2(\mathbb{R}^g) \arrow[d] \\
\mathcal{H}(\Sigma) \arrow[ru, squiggly] & C_k(\Sigma)
\end{tikzcd}$$

bundle of Hilbert spaces

$\xrightarrow{H_k^{BM}(-,\partial_0)}$ Unitary representation of $\widetilde{\mathcal{M}}(\Sigma)$.

Perspectives

- Linearity

$B_n = \mathcal{M}(\mathbb{D}_n)$ is linear [Bigelow, Krammer]

subgroup of $GL_N(F)$ for some F field, N finite integer

$\mathcal{M}(\Sigma_{g,1})$?? $\quad (g=1: \ \mathcal{M}(\Sigma_{1,1}) \cong B_3)$

$\mathrm{Torelli}(\Sigma_{g,1})$??

- Applications to quantum link invariants ? $\longrightarrow$ via a finite-dim. version of the Schrödinger representation using the mod-N Heisenberg group

---

Thank you for listening!

BONUS

Representations of the Morita subgroup

$$\begin{tikzcd}
 & & & U(L) \arrow[d] & \text{central extension by } S^1 \arrow[l] \\
\mathcal{M}(\Sigma) \arrow[r, "\Phi"] & \mathrm{Aut}^+(\mathcal{H}(\Sigma)) \arrow[r, hook] & \mathrm{Aut}^+(\mathcal{H}(\Sigma)_c) \arrow[r, "T"] \arrow[d, phantom, sloped, "\cong"] & PU(L) & \\
 & & Sp_{2g}(\mathbb{R}) \ltimes H_1(\Sigma;\mathbb{R}) & & \\
\mathrm{Morita}(\Sigma) \arrow[uu, hook] \arrow[rr, "\text{action on } H_1(\Sigma;\mathbb{R})"] & & Sp_{2g}(\mathbb{R}) \arrow[u, hook] \arrow[uur] & &
\end{tikzcd}$$

① The pullback of $\begin{matrix} U(L) \\ \downarrow \\ PU(L) \end{matrix}$ to $Sp_{2g}(\mathbb{R})$ contains an embedded copy of $\begin{matrix} Mp_{2g}(\mathbb{R}) \\ \downarrow \\ Sp_{2g}(\mathbb{R}) \end{matrix}$

central ext. by S^1

metaplectic group

unique connected double covering (a $\mathbb{Z}/2$ - central ext.)

$\pi_1 \cong \mathbb{Z}$

② The pullback of $\begin{matrix} Mp_{2g}(\mathbb{R}) \\ \downarrow \\ Sp_{2g}(\mathbb{R}) \end{matrix}$ to $\mathrm{Morita}(\Sigma)$ is <u>trivial</u>.

③ Hence there exists a <u>lift</u>

$$\begin{tikzcd}
 & & & U(L) \arrow[d] \\
\mathrm{Morita}(\Sigma) \arrow[r, hook] \arrow[urrr] & \mathcal{M}(\Sigma) \arrow[rr, "T\circ\Phi"] & & PU(L)
\end{tikzcd}$$

and we may complete the construction as above, but <u>without</u> having to pass to a central extension of $\mathrm{Morita}(\Sigma)$.

Note

- It is <u>not</u> true that the restriction of $\begin{matrix} \mathbb{Z} \\ \downarrow \\ \widetilde{\mathcal{M}}(\Sigma) \\ \downarrow \\ \mathcal{M}(\Sigma) \end{matrix}$ to $\mathrm{Morita}(\Sigma)$ is trivial, although it <u>is</u> when reduced modulo 12 (or any factor of 12).

- To prove ② we show that the pullback of $\begin{matrix} \mathbb{Z}/2 \\ \downarrow \\ Mp_{2g}(\mathbb{R}) \\ \downarrow \\ Sp_{2g}(\mathbb{R}) \end{matrix}$ to $\mathrm{Morita}(\Sigma)$ is isomorphic to $\widetilde{\mathrm{Morita}(\Sigma)}^{(2)}$, the reduction modulo 2 of the restriction of $\widetilde{\mathcal{M}}(\Sigma)$ to $\mathrm{Morita}(\Sigma)$.