Spectrification of Khovanov homology I

GeMAT seminar
14 March 2022

Plan

(today) I The original construction of Lipshitz – Sarkar

(24 March) II A second, simpler construction (Lawson-Lipshitz – Sarkar, Hu-Kriz-Kriz)

(8 April) III An extension to $\underbrace{\text{tangles}}_{(0,1)-\text{TQFT}}$ and $\underbrace{\text{tangle cobordisms}}_{(0,1,2)-\text{TQFT}}$

1. What are Khovanov spectra?
2. Stable cohomology operations
3. Examples, alternating links
4. Spatial lifts
5. Lax spatial lifts
6. Framed flow categories
7. The Khovanov framed flow category } Next time, described in parallel with LLS's simpler construction.

1. Khovanov spectra

Khovanov (2000)

$$L \longmapsto Kh_R(L) = \bigoplus_{i,j} Kh_R^{\overset{\text{homological grading}}{i}, \overset{\text{quantum grading}}{j}}(L)$$

bigraded abelian group (R-module)

such that

$$\chi\left(Kh_R^{*,j}(L)\right) = \text{coefficient of } q^j \text{ in } (q+q^{-1})\underbrace{J(L)}_{\text{Jones polynomial}}$$

$$\| $$

$$\sum_i (-1)^i \operatorname{rank}\left(Kh^{i,j}(L)\right)$$

Lipshitz - Sarkar (2014)

$$L \longmapsto \text{sequence of } \underbrace{\text{suspension spectra}}_{\text{Def: soon!}} \quad X^j_{Kh}(L), \quad j \in \mathbb{Z}$$

such that

$$\widetilde{H}^i\left(X^j_{Kh}(L); R\right) \cong Kh_R^{i,j}(L)$$

Thus:

$$\chi\left(X^j_{Kh}(L)\right) = \text{coefficient of } q^j \text{ in } (q+q^{-1})J(L)$$

More precisely:

Notation:

$\mathbb{Z}[S]$ = sequences of objects of type S, indexed by $j \in \mathbb{Z}$, all but finitely many of them trivial.

e.g. $\mathbb{Z}[\mathbb{Z}]$ = Laurent polynomials in q

viewing the j^{th} integer as the coeff. of q^j

$$
\begin{tikzcd}
\{\text{link diagrams}\} \arrow[r] \arrow[dd] \arrow[rr, bend left] \arrow[rrrd, bend left] & \mathbb{Z}\left[\begin{smallmatrix}\text{finite}\\ \text{CW-complexes}\end{smallmatrix}\right] \arrow[r, "\tilde{C}^*_{cell}"] \arrow[dd, "\Sigma^\infty"] & \mathbb{Z}\left[\begin{smallmatrix}\text{chain}\\ \text{complexes}\end{smallmatrix}\right] \arrow[dd, "H_*"] \arrow[dr, "\chi"] & \\
 & & & \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[q^{\pm 1}] \\
\{\text{links}\} \arrow[r, "X_{Kh}"] \arrow[rr, bend right, "Kh"'] \arrow[rrru, bend right, "J"'] & \mathbb{Z}\left[\begin{smallmatrix}\text{finite suspension}\\ \text{spectra up to } \simeq\end{smallmatrix}\right] \arrow[r, "\tilde{H}^*"'] & \mathbb{Z}\left[\begin{smallmatrix}\text{graded}\\ \text{modules}\end{smallmatrix}\right] \arrow[ur, "\chi"'] &
\end{tikzcd}
$$

J = Jones polynomial

Kh = Khovanov homology

X_{Kh} = Lipshitz-Sarkar-Khovanov spectrum

Suspension spectra

- $\text{Spaces} \xrightarrow{\Sigma^\infty} \text{Spectra}$

  complicated to define

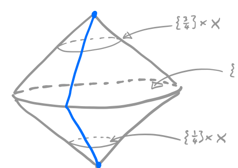
- The subcategory on the objects $\Sigma^\infty X$, $X =$ finite CW-complex, has a simple description. This is the category of finite suspension spectra.

- Objects (X, i) for $\begin{cases} X & \text{finite based CW-complex} \\ i \in \mathbb{Z} \end{cases}$

  written $\Sigma^i X$

  identify $\Sigma^{i+1} X = \Sigma^i(\underbrace{\Sigma X})$

  (reduced) suspension of X $= \dfrac{[0,1] \times X}{(\{0,1\} \times X) \cup ([0,1] \times \{*\})}$

  $=$ 

  with | collapsed to a point

- Morphisms $\Sigma^i X \to \Sigma^j Y$ are $(k, \Sigma^{i+k} X \xrightarrow{f} \Sigma^{j+k} Y)$

  such that $i+k$ and $j+k \geq 0$

  usual based map of spaces

  identify $(k, f) \sim (k+1, \Sigma f)$

- Homotopies (2-morphisms) defined similarly.

- $\Sigma^\infty : (\text{finite CW-complexes}) \longrightarrow (\text{finite suspension spectra})$

  $$X \longmapsto \Sigma^0 X$$

- **Upshot**: $\Sigma^\infty X \simeq \Sigma^\infty Y$ if and only if $\exists k : \Sigma^k X \simeq \Sigma^k Y$.

(2) Stable cohomology operations

- Spectra contain much more information than their cohomology groups $H^i(X;R)$.

- How to extract this information as algebraic invariants?

X spectrum $\rightsquigarrow$ $\tilde{H}^*(X;\mathbb{F}_p)$ is a module over the <u>Steenrod algebra</u> $\mathcal{A}_p$

- graded $\mathbb{F}_p$-algebra
- known explicit presentation

- other generalised cohomology theories

  e.g. (topological) K-theory $K^*(X)$

- which come with their own algebras of stable cohomology operations

  (e.g. Adams operations for K-theory)

<u>$\mathcal{A}_2$</u>

- generated by Sq^n for $n \geq 1$. $\quad Sq^n : \tilde{H}^i(X;\mathbb{F}_2) \longrightarrow \tilde{H}^{i+n}(X;\mathbb{F}_2)$
- subject to the <u>Adem relations</u>

- e.g. the subalgebra generated by $\{Sq^1, Sq^2\}$ is

$$\mathcal{A}_2(1) = \mathbb{F}_2\{Sq^1, Sq^2\} \Big/ \begin{pmatrix} Sq^1Sq^1 = 0 \\ Sq^2Sq^2 = -Sq^1Sq^2Sq^1 \end{pmatrix}$$

Rmks

(1) No nontrivial stable coh. operations on $\widetilde{H}^*(X;\mathbb{Q})$.

(2) Stable coh. operations on $\widetilde{H}^*(X;\mathbb{Z})$ are <u>determined</u> by the stable coh. operations on $\widetilde{H}^*(X;\mathbb{F}_p)$ for all primes p.

(3) The cohomology $\widetilde{H}^*(X)$ of <u>spaces</u> X also has

- a cup product
- Massey products, etc.

BUT these do not exist for spectra.

---

(4) Atiyah-Hirzebruch spectral sequence

$$H^*(X; E^*(pt)) \Rightarrow E^*(X)$$

any generalised cohomology theory

<u>Fact</u>: d_2, d_3 differentials for $E^* =$ K-theory are determined by Sq^1 and Sq^2 operations.

So sometimes $Kh(L)$ plus Sq^1 and Sq^2 determine <u>Khovanov K-theory</u>

$$K^*(X^j_{Kh}(L))$$

[LS] compute this for all links with ≤ 11 crossings, and it is just $Kh^{*,j}(L) \otimes \underbrace{K^*(pt)}_{\begin{cases}\mathbb{Z} & \text{even degrees} \\ 0 & \text{odd degrees}\end{cases}}$.

But
- there may be non-trivial Adams operations
- in general $K^*(X^j_{Kh}(L))$ is expected to be richer than $Kh^{*,j}(L)$.

3. Examples, alternating links

Unknot

quantum grading j	Khovanov spectrum $\mathcal{X}^j_{Kh}$
1	S^0
-1	S^0

Hopf link

quantum grading j	Khovanov spectrum $\mathcal{X}^j_{Kh}$
6	S^2
4	S^2
2	S^0
0	S^0

More generally:

Prop [LS] If L is an alternating link, then $\mathcal{X}^j_{Kh}(L)$ is a wedge sum of Moore spaces. In particular:

- it is determined by its (co)homology
- all Steenrod operations are trivial.

Rmk. In fact, this is true of any spectral refinement of $Kh(L)$, independently of how it is defined.

- It follows from the fact* that $Kh^{i,j}(L) \neq 0$ only when $i = \frac{1}{2}(j + \underbrace{\sigma(L)}_{\text{signature}} \pm 1)$ and has torsion only when $i = \frac{1}{2}(j + \sigma(L) + 1)$.

*due to [Manolescu-Ozsváth, 2007]

BUT

$K = 10_{145}$

quantum grading j	Khovanov spectrum $\mathcal{X}^j_{Kh}$
-3	S^0
-5	S^0
-7	$\Sigma^{-3}(S^0 \vee S^1)$
-9	$\Sigma^{-6}(\mathbb{RP}^2 \wedge \mathbb{RP}^2)$
-11	$\Sigma^{-5}(S^0 \vee S^1 \vee S^1 \vee S^2)$
-13	$\Sigma^{-8}(\mathbb{RP}^4/\mathbb{RP}^1 \vee \Sigma \mathbb{RP}^2)$
-15	$\Sigma^{-10}(\mathbb{RP}^5/\mathbb{RP}^2 \vee S^4)$
-17	$\Sigma^{-8}(S^0 \vee S^1)$
-19	$\Sigma^{-10}\mathbb{RP}^2$
-21	$\Sigma^{-9} S^0$

Theorem [Seed] $\exists\, L_1, L_2$ with $Kh(L_1) \cong Kh(L_2)$

but $\mathcal{X}_{Kh}(L_1) \not\simeq \mathcal{X}_{Kh}(L_2)$

shown by computing Sq^2

$$\begin{pmatrix} L_1 = 11^n_{70} \\ L_2 = 13^n_{2566} \end{pmatrix}$$

Theorem [LLS] $\forall n \geq 1$, $\exists$ knot L such that

$$Sq^n : Kh^{*,*}_{\mathbb{F}_2}(L) \longrightarrow Kh^{*+n,*}_{\mathbb{F}_2}(L)$$

is non-trivial.

Explicit combinatorial formulas for $Sq^n : Kh^{*,*}_{\mathbb{F}_2}(L) \longrightarrow Kh^{*+n,*}_{\mathbb{F}_2}(L)$

- Lipshitz-Sarkar $n = 1, 2, 3$
- Cantero all n

4. Spatial lifts

Aim: Idea of how to lift to

$$
\begin{tikzcd}
\{\text{link diagrams}\} \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathbb{Z}\left[\begin{array}{c}\text{finite}\\ \text{CW-complexes}\end{array}\right] \arrow[r, "\widetilde{C}^*_{\text{cell}}"] \arrow[d, "\Sigma^\infty"] & \mathbb{Z}\left[\begin{array}{c}\text{chain}\\ \text{complexes}\end{array}\right] \arrow[d, "H_*"] \\
\{\text{links}\} \arrow[r, "\mathcal{X}_{Kh}"] \arrow[rr, bend right, "Kh"'] & \mathbb{Z}\left[\begin{array}{c}\text{finite suspension}\\ \text{spectra up to } \simeq\end{array}\right] \arrow[r, "\widetilde{H}^*"'] & \mathbb{Z}\left[\begin{array}{c}\text{graded}\\ \text{modules}\end{array}\right]
\end{tikzcd}
$$

Then one has to check that the (sequence of) CW-complexes associated to a link diagram is invariant <u>up to stable $\simeq$</u> under Reidemeister moves.

Fix a quantum grading j. Then we want to lift

$$
\begin{tikzcd}
 & \{\text{CW-complexes}\} \arrow[d, "\widetilde{C}^*_{\text{cell}}"] \\
\{\text{link diagrams}\} \arrow[ur, dashed] \arrow[r, "KhC^{*,j}"'] & \{\text{chain complexes}\}
\end{tikzcd}
$$

Def $B(\mathbb{Z})$ $\begin{cases} \text{objects} = \text{finite sets} \\ \text{maps } X \to Y = \text{homomorphisms } \bigoplus_X \mathbb{Z} \to \bigoplus_Y \mathbb{Z} \end{cases}$

$B(S^N)$ $\begin{cases} \text{objects} = \text{finite sets} \\ \text{maps } X \to Y = \text{continuous maps } \bigvee_X S^N \to \bigvee_Y S^N \end{cases}$

- A $\mathbb{Z}$-chain complex is a linear diagram in $B(\mathbb{Z})$:

$$F(n) \to \cdots \to F(1) \to F(0)$$

such that any pair composes to the zero homomorphism.

Note: By abuse of notation, we conflate a finite set X with the free abelian group $\bigoplus_X \mathbb{Z}$ that it generates or the wedge of spheres $\bigvee_X S^N$ that it indexes, depending on the context.

- An S^N-chain complex is a linear diagram in $B(S^N)$:

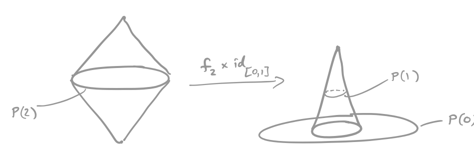
$$P(n) \xrightarrow{f_n} \cdots \xrightarrow{f_2} P(1) \xrightarrow{f_1} P(0)$$

such that any pair composes to the constant map.

A strict spatial lift of a chain complex F is an S^N-chain complex P such that $\underbrace{H_N}_{\text{degree-}N\text{ homology}} \circ P = F$.

From this we can construct a CW-complex X_P with $\widetilde{C}^*(X_P) \cong F$ as follows.

The fact that $f_1 \circ f_2 = *$ means that $\exists$ map $g_1 : \Sigma P(2) \longrightarrow \mathrm{Cone}(f_1)$



Thus we obtain a new sequence of CW-complexes with each pair of maps composing to the constant map:

$$\Sigma P(n) \xrightarrow{\Sigma f_n} \cdots \longrightarrow \Sigma P(3) \xrightarrow{\Sigma f_3} \Sigma P(2) \xrightarrow{g_1} \mathrm{Cone}(f_1)$$

Iterating, we eventually obtain a CW-complex X_P whose reduced cellular cochain complex is the original chain complex F.

<u>Summary</u>:

$$\begin{tikzcd}
\{S^N\text{-chain complexes}\} \arrow[r, "P \mapsto X_P"] \arrow[dr, "H_N"'] & \{\text{CW-complexes}\} \arrow[d, "\tilde{C}^*"] \\
& \{\mathbb{Z}\text{-chain complexes}\}
\end{tikzcd}$$

So it would be enough to refine the Khovanov chain complex to an S^N-chain complex.

<u>BUT</u>:

- $B(\mathbb{Z})(X,Y) = \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_X \mathbb{Z}, \bigoplus_Y \mathbb{Z}\right) = (Y \times X)$-matrices over $\mathbb{Z}$

  (because $\oplus$ is a product and coproduct)

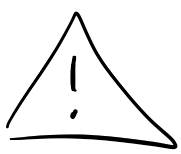
- $B(S^N)(X,Y) = \mathrm{Map}\left(\bigvee_X S^N, \bigvee_Y S^N\right)$

  $= \prod_X \mathrm{Map}\left(S^N, \bigvee_Y S^N\right)$ (because $\vee$ is a coproduct)

$$\pi_0 \mathcal{B}(S^N)(X,Y) = \bigoplus_{X} \underbrace{\pi_N\Big(\bigvee_{Y} S^N\Big)}_{\bigoplus_{Y} \pi_N(S^N)} \cong \bigoplus_{X} \bigoplus_{Y} \pi_N(S^N) \cong \mathcal{B}(\mathbb{Z})(X,Y).$$

exercise, or
Hilton-Milnor theorem

So a strict spatial lift P of F does not contain any more information (up to $\simeq$) than F. Hence the homotopy type of X_P also cannot contain any more information.



5. Lax spatial lifts

A **lax S^N-chain complex** is a linear diagram

$$P(n) \xrightarrow{f_n} \cdots \xrightarrow{f_2} P(1) \xrightarrow{f_1} P(0),$$

where $P(n) = \bigvee_{F(n)} S^N$, together with

- for each pair
$$\begin{tikzcd} \bigvee S^N \arrow[r, "f"] \arrow[rr, bend right, "H", "*"'] & \bigvee S^N \arrow[r, "g"] & \bigvee S^N \end{tikzcd}$$
a homotopy

- for each triple
$$\begin{tikzcd} \bigvee S^N \arrow[r, "f"] \arrow[rr, bend right, "H", "*"'] & \bigvee S^N \arrow[r, "g"] \arrow[rr, bend right, "K", "*"'] & \bigvee S^N \arrow[r, "h"] & \bigvee S^N \end{tikzcd}$$
a homotopy of homotopies:
$$\begin{tikzcd} * \arrow[r, dash, "*"] \arrow[d, dash, "hH"'] & * \arrow[d, dash, "*"] \\ hgf \arrow[r, dash, "Kf"'] \arrow[ur, phantom, "\Phi"] & * \end{tikzcd} \times \bigvee S^N \longrightarrow \bigvee S^N$$

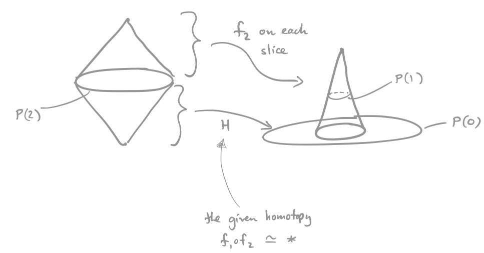
- etc.

Emphasis: The maps f_i, homotopies $H, K, \ldots$, higher homotopies $\Phi, \ldots$, etc. are all choices — i.e. they are part of the data of a lax S^N-chain complex.

A **lax spatial lift** of a chain complex F is a lax S^N-chain complex P such that $H_N \circ P = F$.

A lax spatial lift P also determines a CW-complex X_P whose reduced cellular chain complex is the original chain complex F.

- Similar construction to above (iterative), except now the map $g_1 : \Sigma P(2) \longrightarrow \mathrm{Cone}(f_1)$ is defined by



- Formally it can be defined as the homotopy colimit of a certain diagram.

Thus we have :

$$\begin{tikzcd}
\{\text{lax } S^N\text{-chain complexes}\} \arrow[r, "P \mapsto X_P"] \arrow[dr, "H_N"'] & \{\text{CW-complexes}\} \arrow[d, "\tilde{C}^*"] \\
& \{\mathbb{Z}\text{-chain complexes}\}
\end{tikzcd}$$

So we need to construct a lax spatial lift of the Khovanov chain complex.

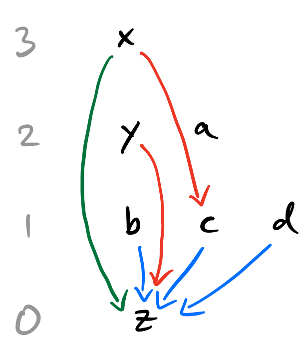
6. Framed flow categories

$$\begin{tikzcd}
\left\{\begin{matrix}\text{framed flow}\\ \text{categories}\end{matrix}\right\} \arrow[r, "\text{Cohen-Jones-Segal}"] \arrow[drr] & \{\text{lax } S^N\text{-chain complexes}\} \arrow[r, "P \mapsto X_P"] \arrow[dr, "H_N"] & \{\text{CW-complexes}\} \arrow[d, "\tilde{C}^*"] \\
\{\text{link diagrams}\} \arrow[u, "\text{Lipshitz-Sarkar}"] \arrow[rr, "KhC^{*,j}"'] & & \{\mathbb{Z}\text{-chain complexes}\}
\end{tikzcd}$$

A framed flow category has a finite set of objects, graded by $\{0, \dots, n\}$, and for objects x, y:

- $\mathcal{M}(x,y)$ k-dimensional manifold with corners,
  $$k = gr(x) - gr(y) - 1$$
- composition is an inclusion
  $$\mathcal{M}(x,y) \times \mathcal{M}(y,z) \hookrightarrow \partial\mathcal{M}(x,z).$$
- framed embeddings of all of these manifolds into high-dim. Euclidean space.

Picture:



$\rightarrow$ 0-dim space of morphisms

$\rightarrow$ 1-dim space of morphisms

$\rightarrow$ 2-dim space of morphisms

Idea:

- $\{$framed flow categories$\} \longrightarrow \{\mathbb{Z}$-chain complexes$\}$

  - Objects $\rightsquigarrow$ generators
  - Forget all higher-dim. morphism spaces
  - #'s of 0-dim. morphism spaces $\rightsquigarrow$ coefficients of the differential.

- [CJS] construction refines this to a lax S^N-chain complex using the higher-dim. morphism spaces and the Pontrjagin-Thom collapse construction.

- [LS] construct a framed flow category from the Khovanov cube of resolutions...